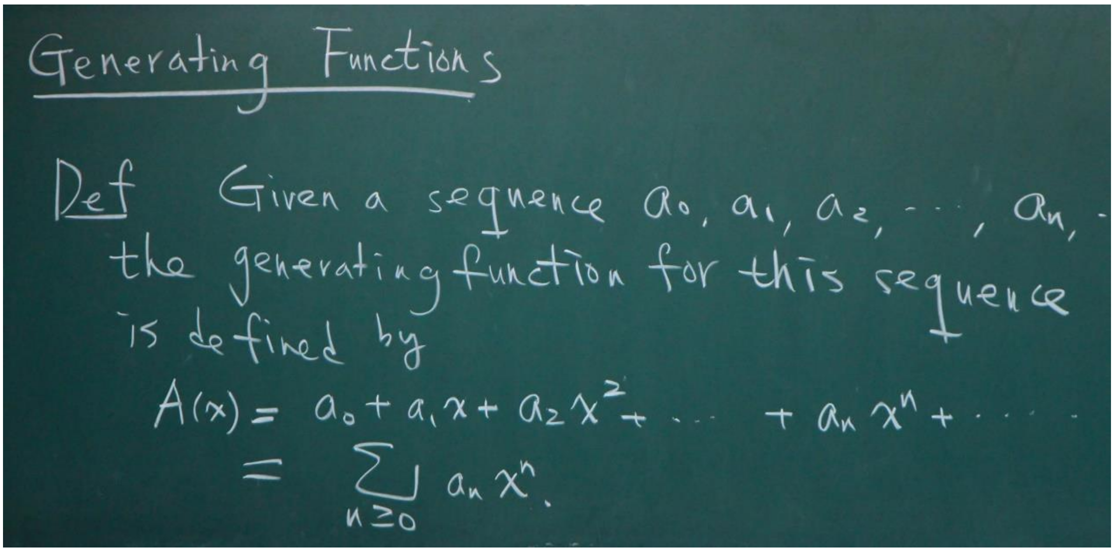
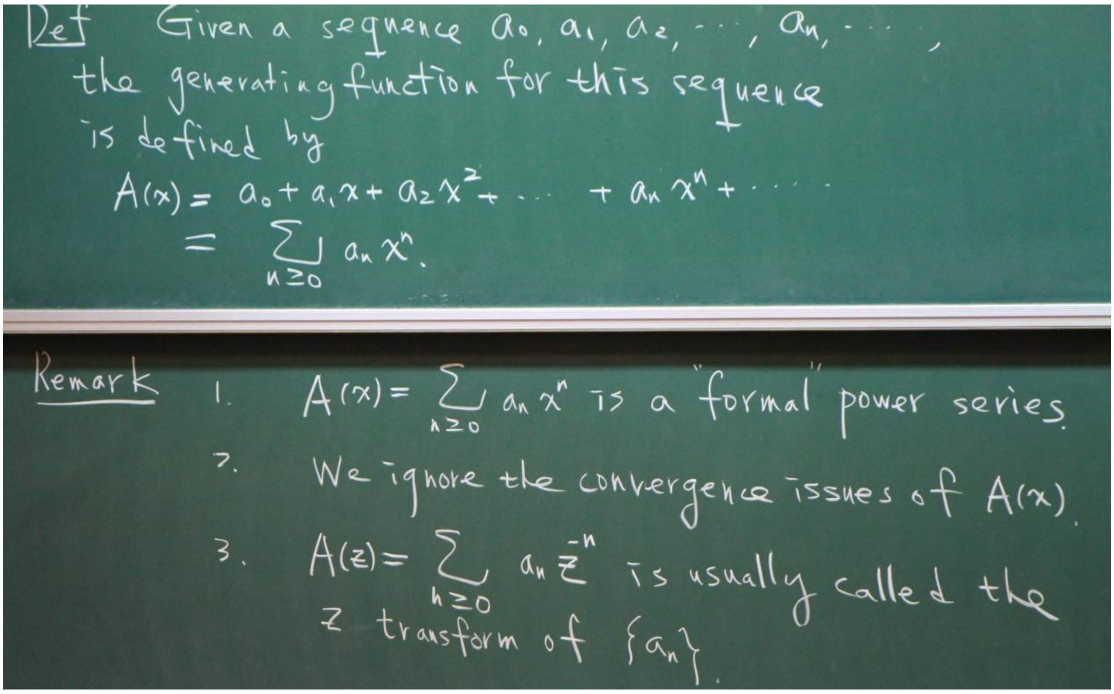
【10920 趙啟超教授離散數學 / 第 14 堂版書】



Generating Functions

Def Given a sequence $a_0, a_1, a_2, \cdots, a_n, \cdots$ the generating function for this sequence is defined by

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots = \sum_{n \ge 0} a_n x^n.$$



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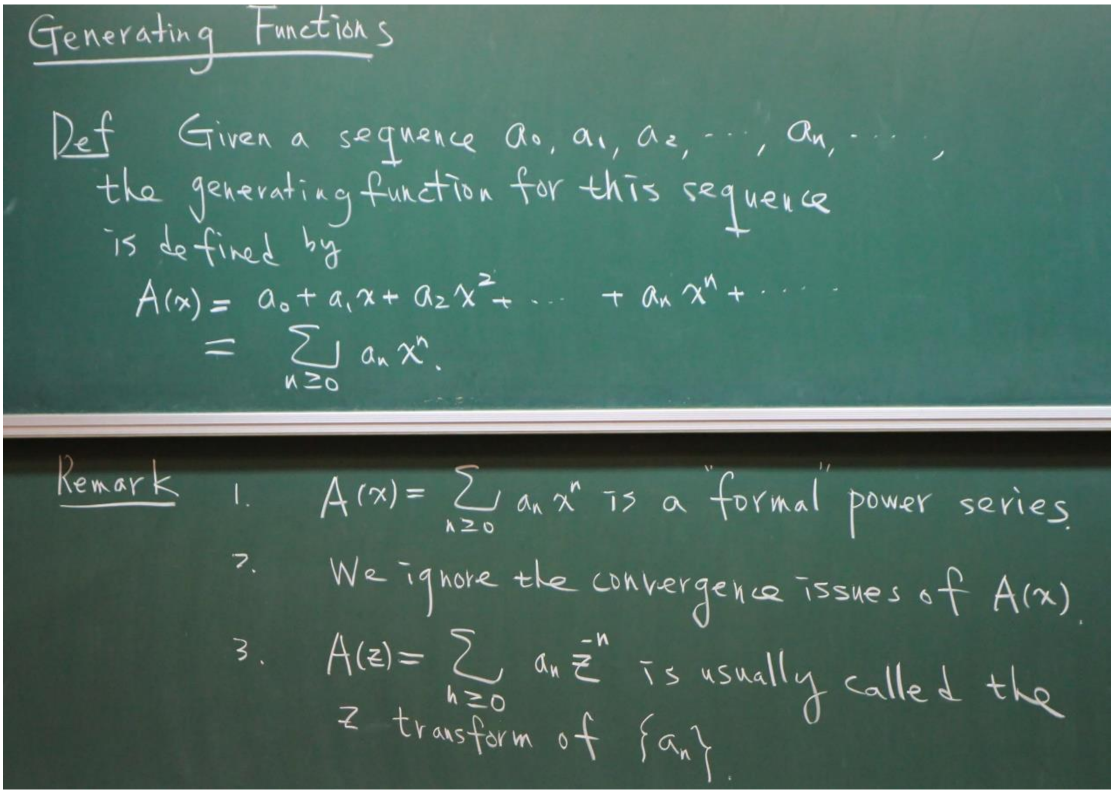
Remark

1. $A(x) = \sum_{n \ge 0} a_n x^n$ is a "formal" power series.
2. We ignore the convergence issues of $A(x)$.
3. $A(z) = \sum_{n \ge 0} a_n z^{-n}$ is usually called the z transform of $\{a_n\}$.



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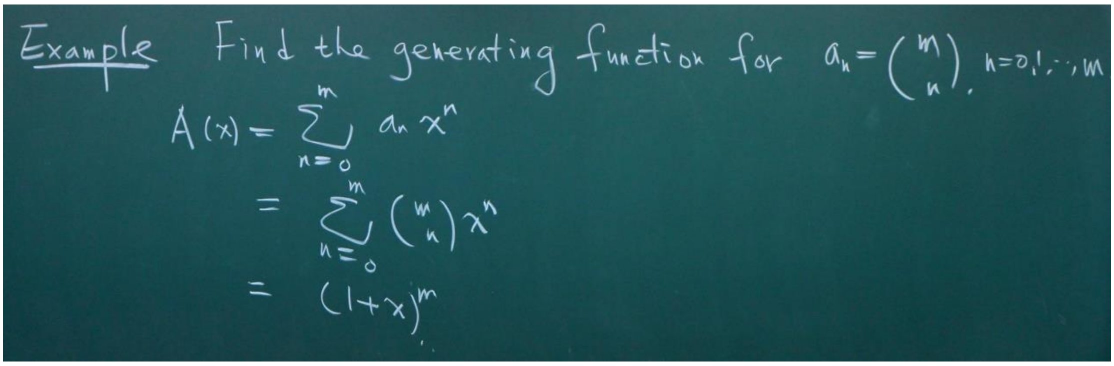

Example Find the generating function for $a_n = \binom{m}{n}$, $n = 0, 1, \cdots, m$

$$A(x) = \sum_{n=0}^{m} a_n x^n$$
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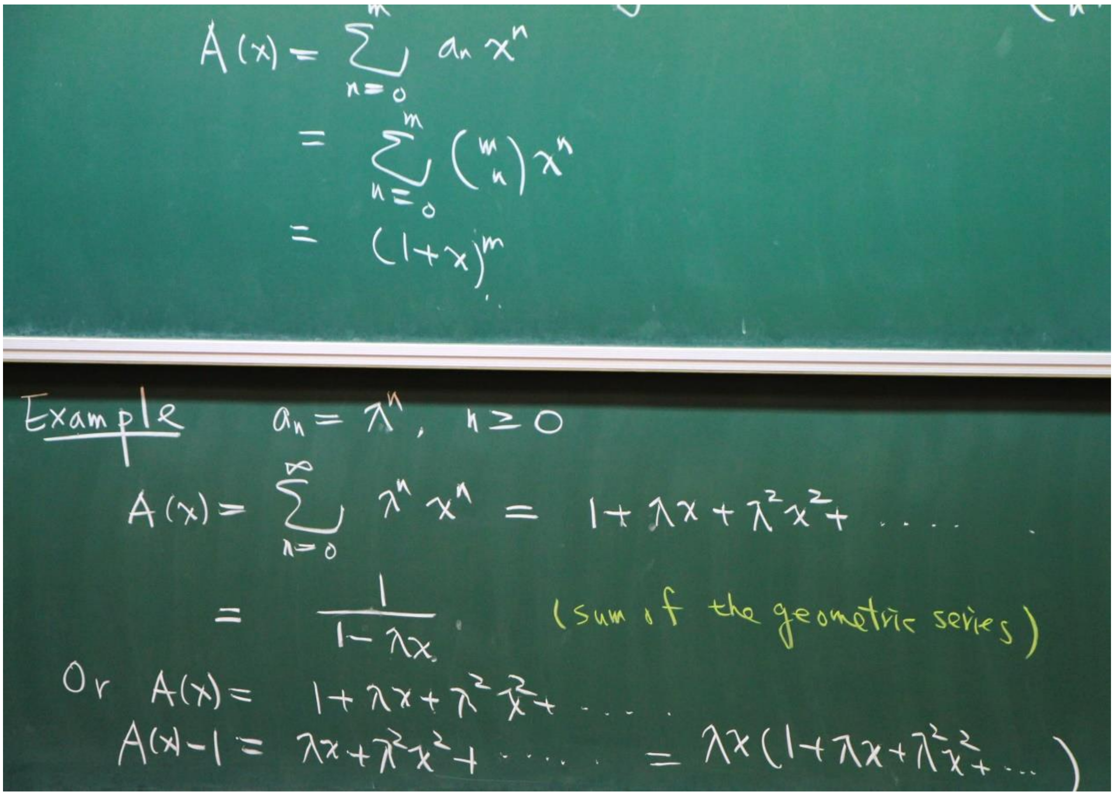
$$= (1+x)^m$$



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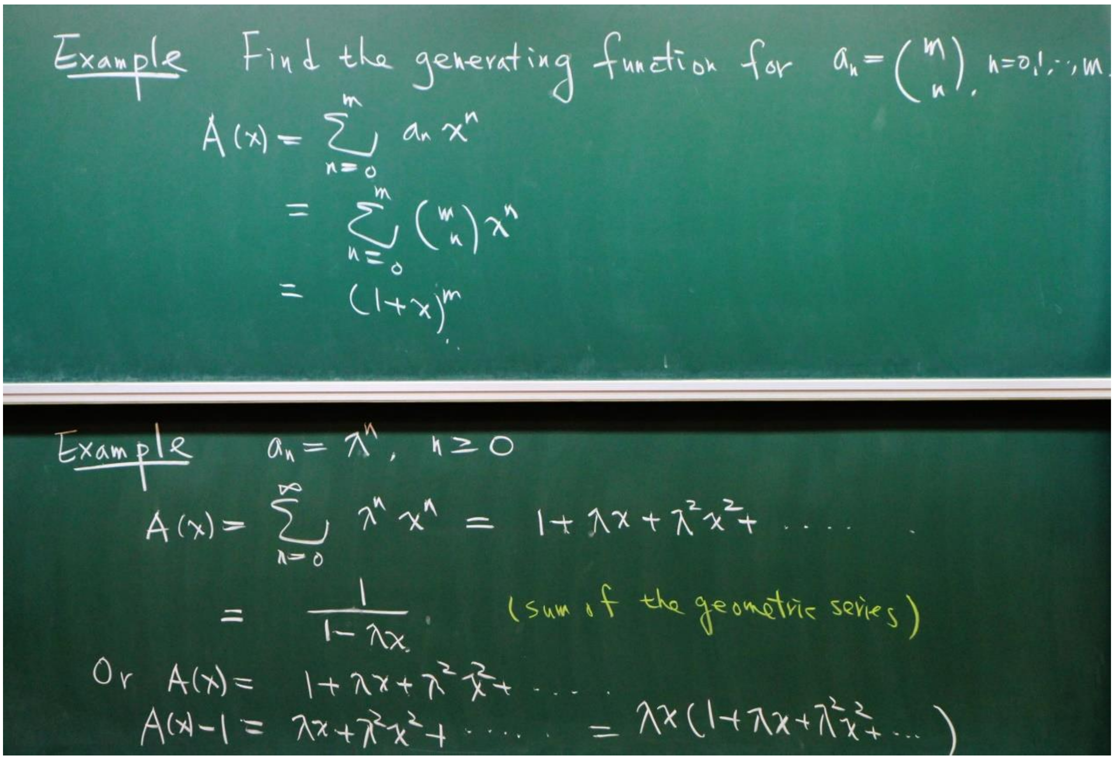
Example $a_n = \lambda^n$, $n \geq 0$

$$A(x) = \sum_{n=0}^{\infty} \lambda^n x^n = 1 + \lambda x + \lambda^2 x^2 + \cdots$$
$$= \frac{1}{1-\lambda x}$$

(sum of the geometric series)

Or $A(x) = 1 + \lambda x + \lambda^2 x^2 + \cdots$

$A(x) - 1 = \lambda x + \lambda^2 x^2 + \cdots = \lambda x (1 + \lambda x + \lambda^2 x^2 + \cdots)$



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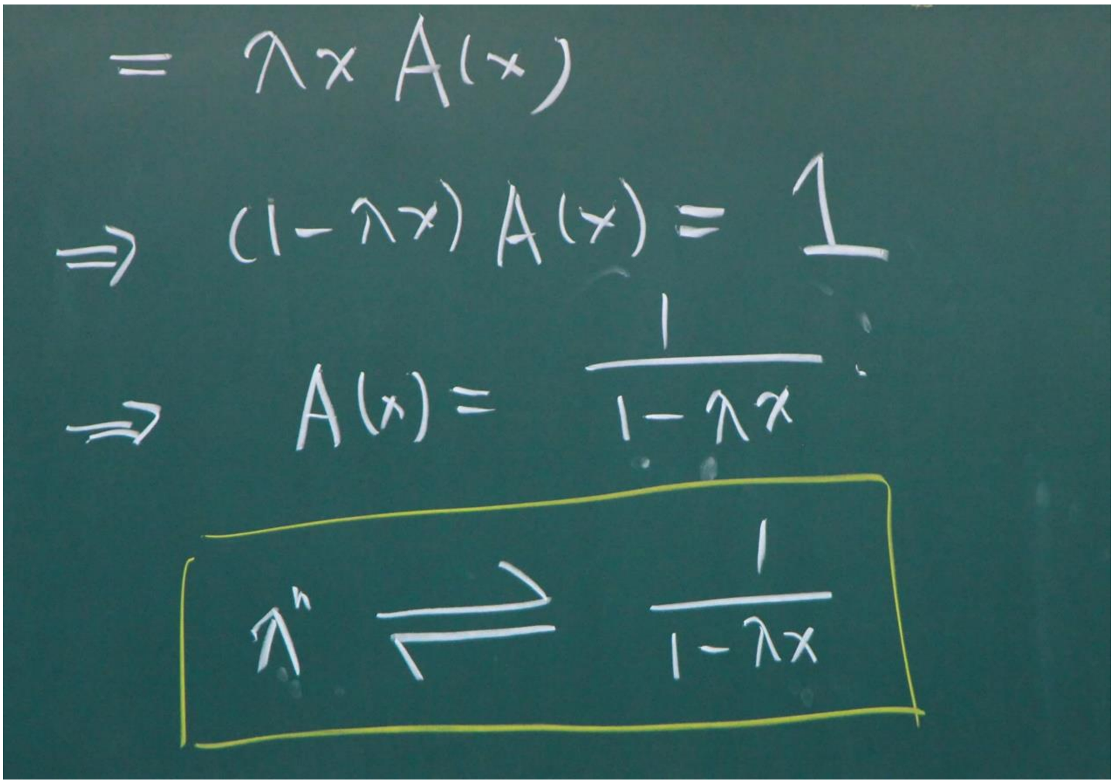
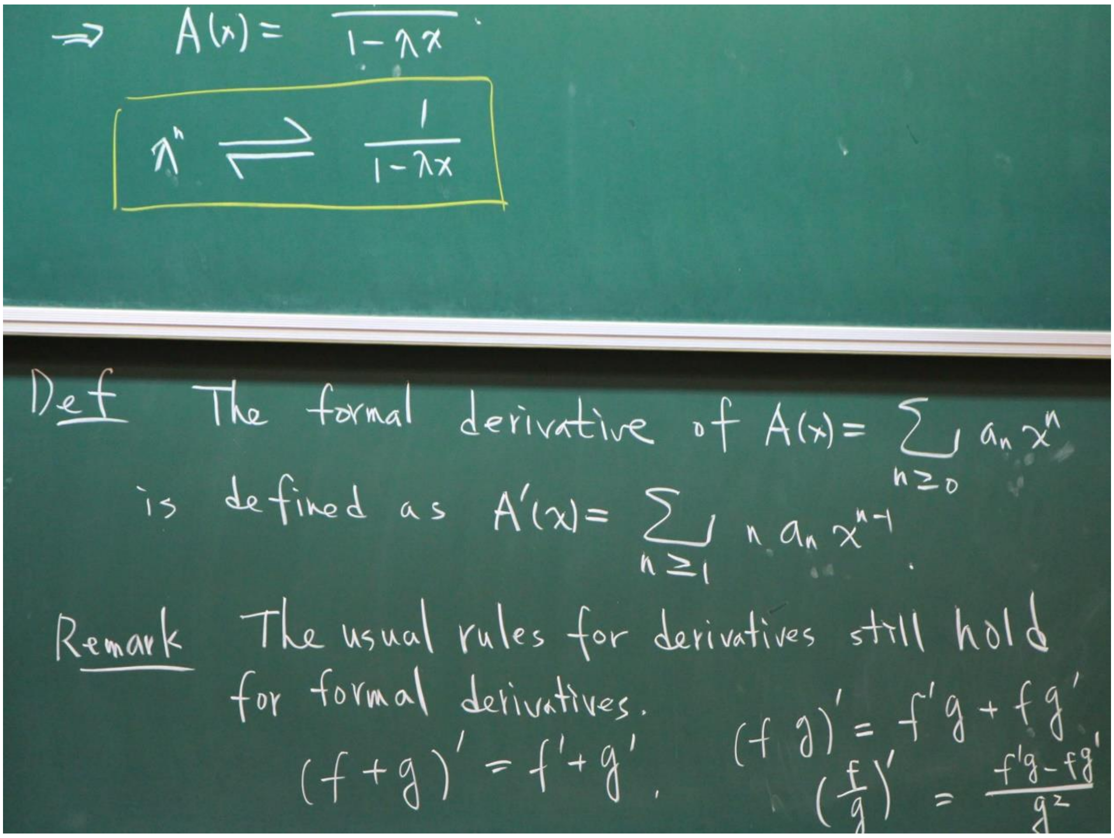


$$= \lambda x A(x)$$

$$\Rightarrow (1-\lambda x) A(x) = 1$$

$$\Rightarrow A(x) = \frac{1}{1-\lambda x}$$

$$\boxed{\lambda^n \rightleftharpoons \frac{1}{1-\lambda x}}$$



$$\Rightarrow A(x) = \overline{1-\lambda x}$$

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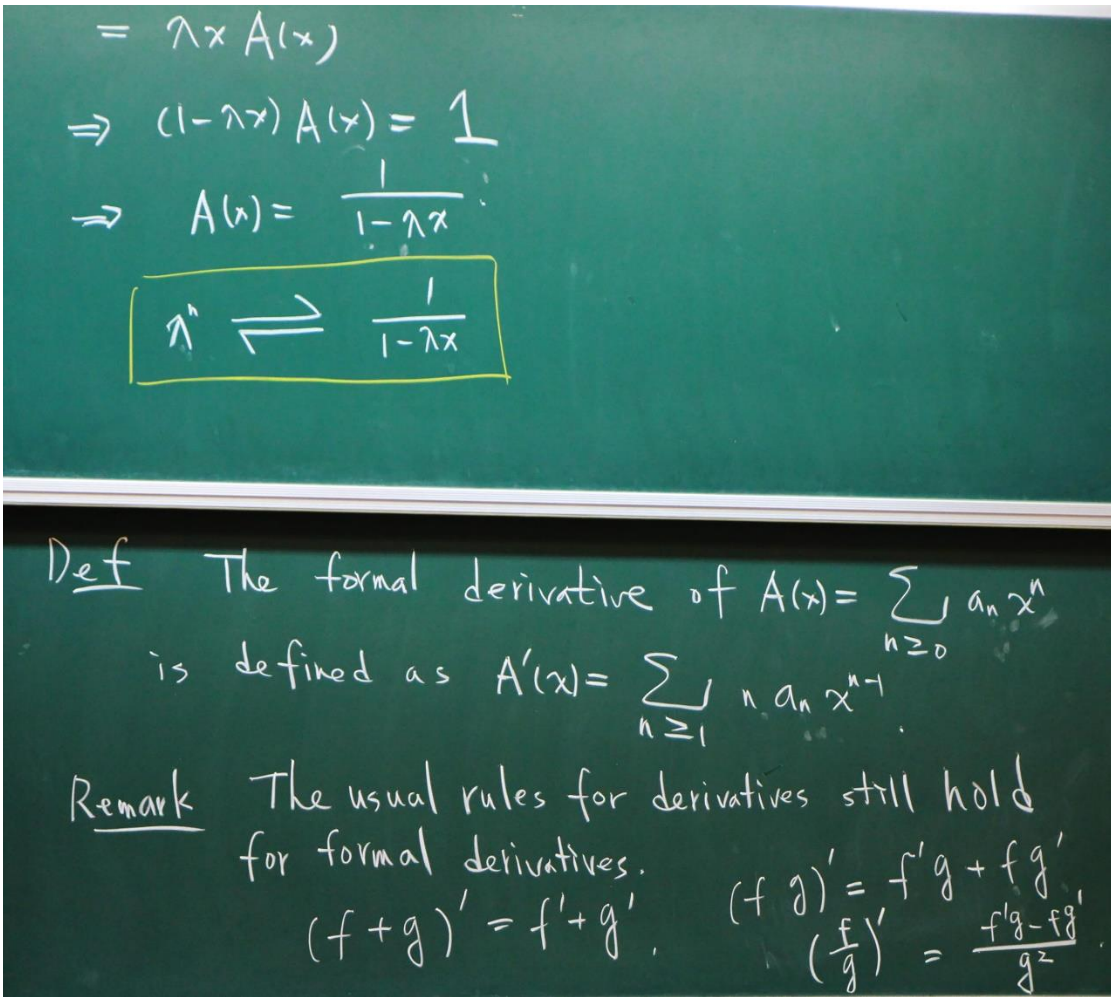
Def The formal derivative of $A(x)=\sum_{n\geq 0} a_n x^n$ is defined as $A'(x)=\sum_{n\geq 1} n\, a_n x^{n-1}$.

Remark The usual rules for derivatives still hold for formal derivatives.

$(f+g)'=f'+g'$, $(fg)'=f'g+fg'$

$\left(\frac{f}{g}\right)' = \frac{f'g-fg'}{g^2}$



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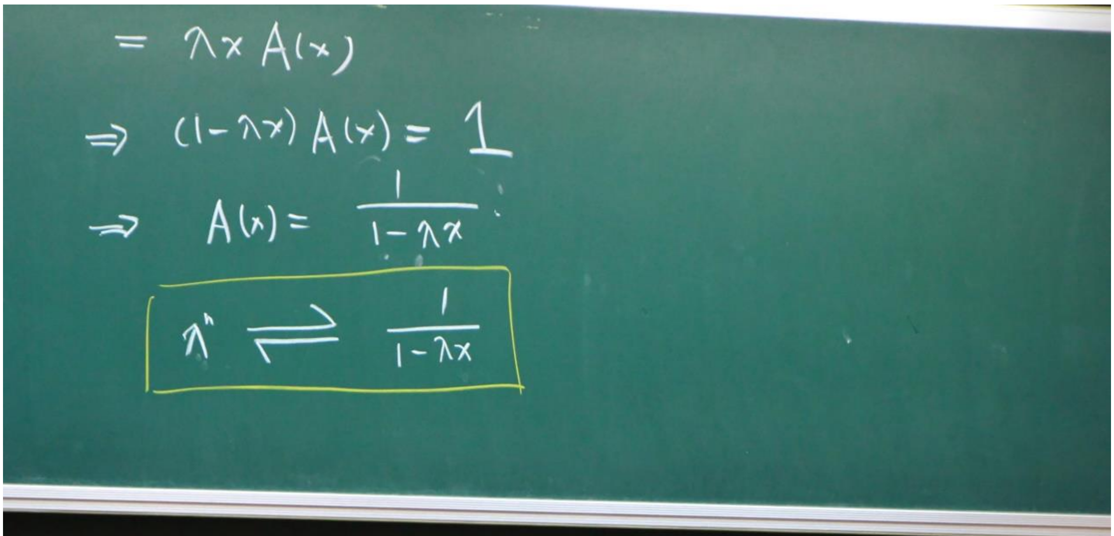

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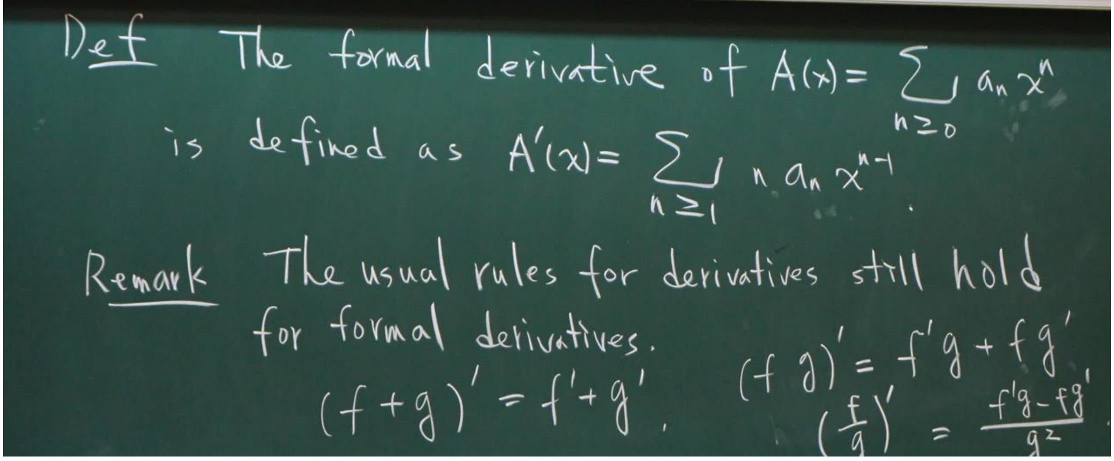

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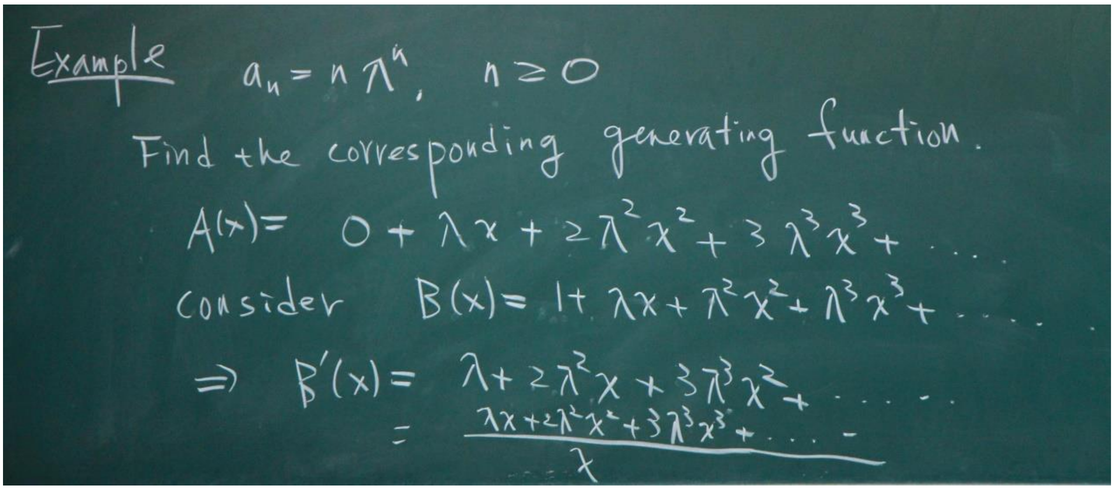


Example $a_n = n\lambda^n$, $n \ge 0$

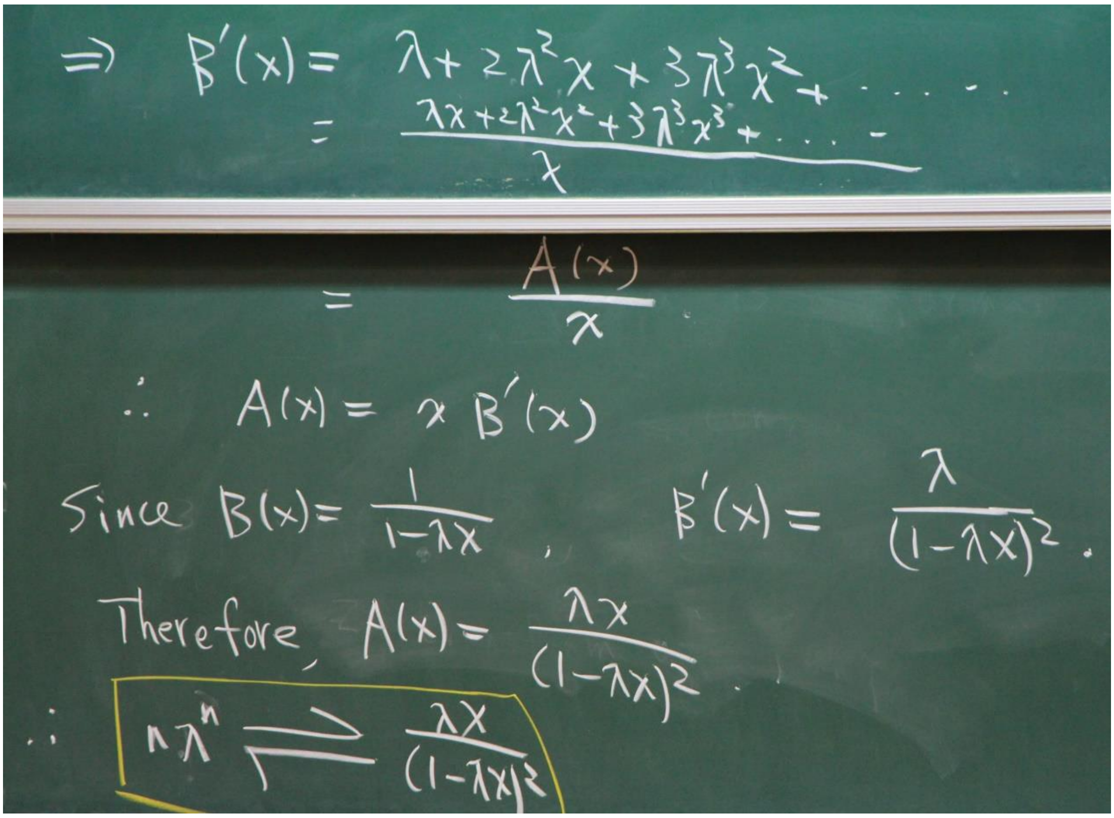
Find the corresponding generating function.

$$A(x) = 0+\lambda x+2\lambda^2x^2+3\lambda^3x^3+\cdots$$

consider $B(x)=1+\lambda x+\lambda^2x^2+\lambda^3x^3+\cdots$

$$\Rightarrow B'(x) = \lambda+2\lambda^2 x+3\lambda^3x^2+\cdots$$

$$= \frac{\lambda x+2\lambda^2x^2+3\lambda^3x^3+\cdots}{x}$$



$$\Rightarrow \quad B'(x) = \lambda + 2\lambda^2 x + 3\lambda^3 x^2 + \cdots\cdots$$

$$= \frac{\lambda x + 2\lambda^2 x^2 + 3\lambda^3 x^3 + \cdots\cdots}{x}$$

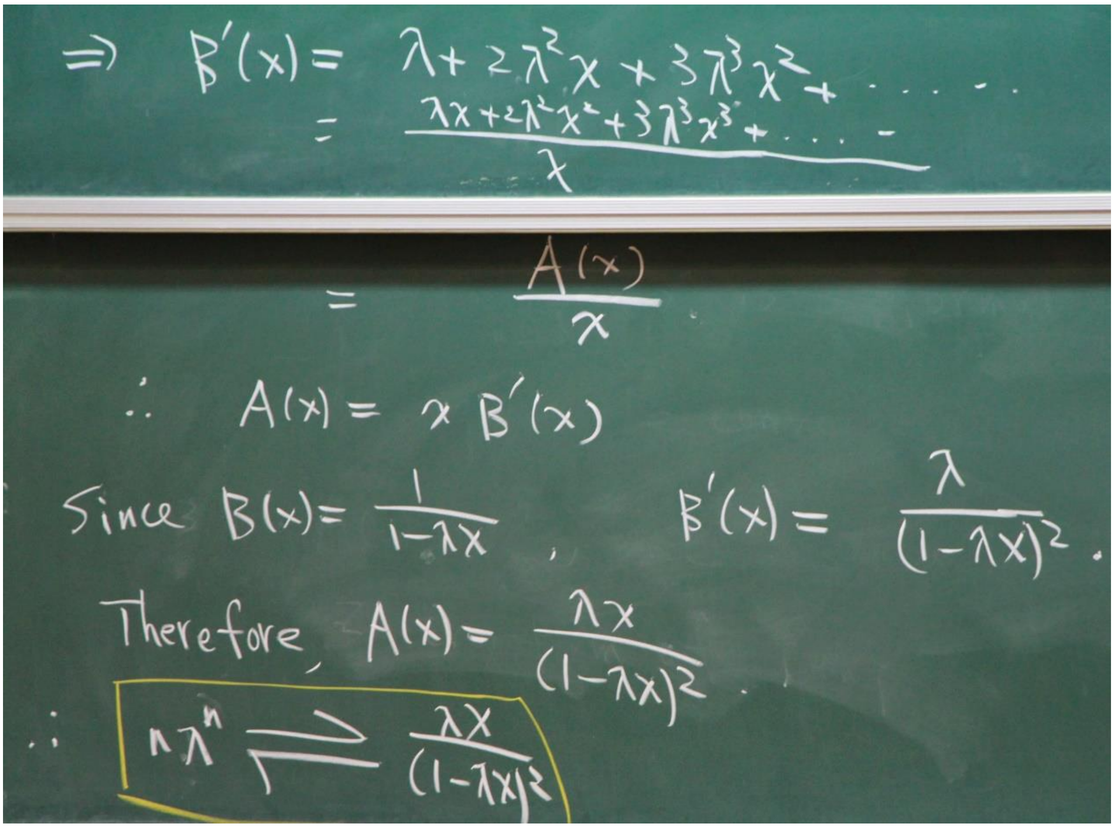
$$= \frac{A(x)}{x}.$$

$$\therefore \quad A(x) = x\,B'(x)$$

Since $B(x) = \frac{1}{1-\lambda x}$, $B'(x) = \frac{\lambda}{(1-\lambda x)^2}$.

Therefore, $A(x) = \frac{\lambda x}{(1-\lambda x)^2}$.

$$\therefore \quad \boxed{n\lambda^n \rightleftharpoons \frac{\lambda x}{(1-\lambda x)^2}}$$



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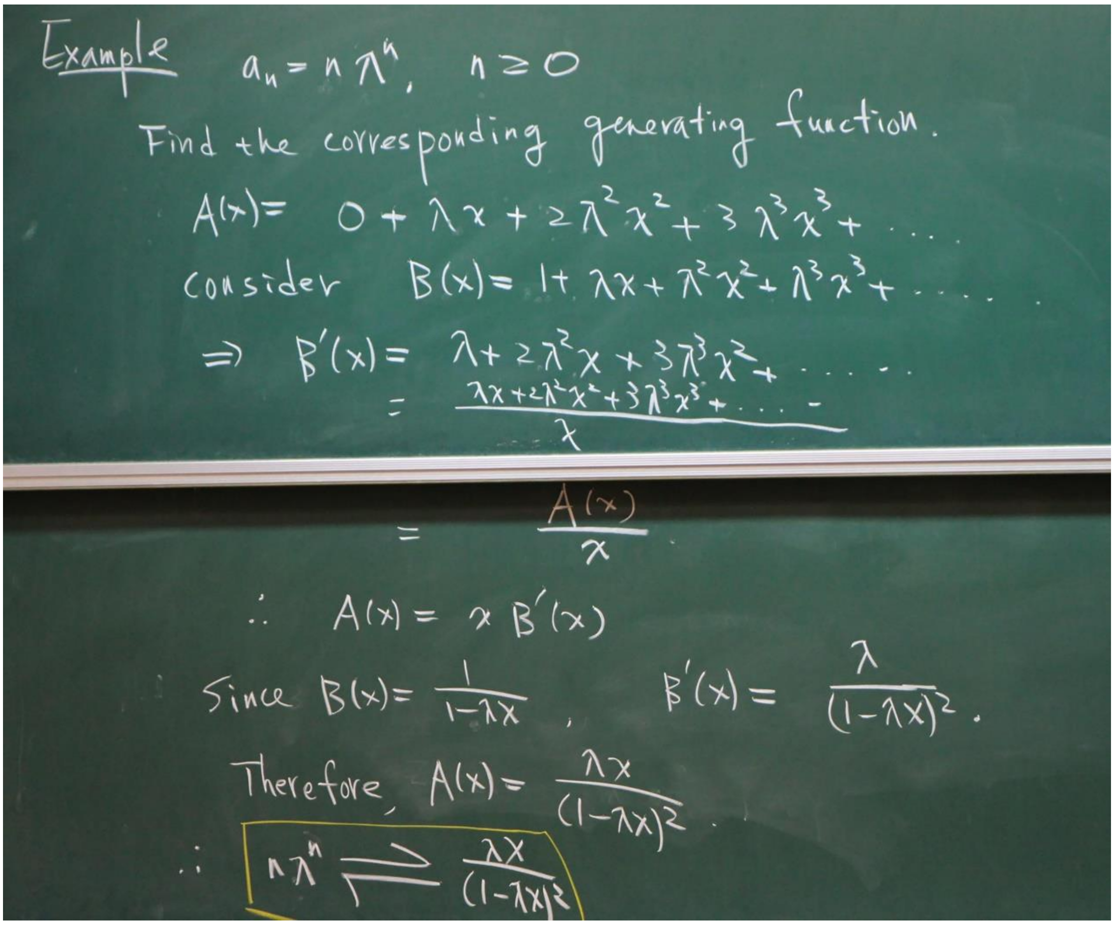

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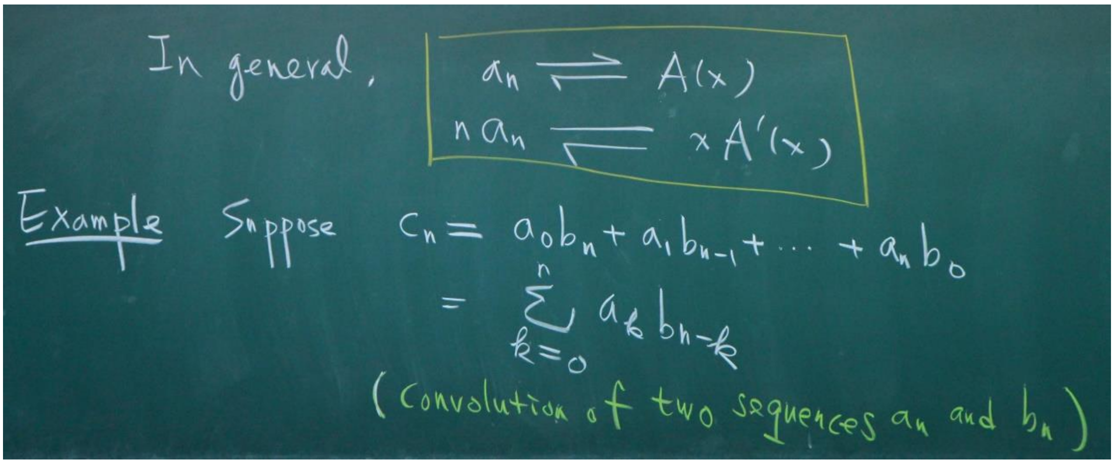
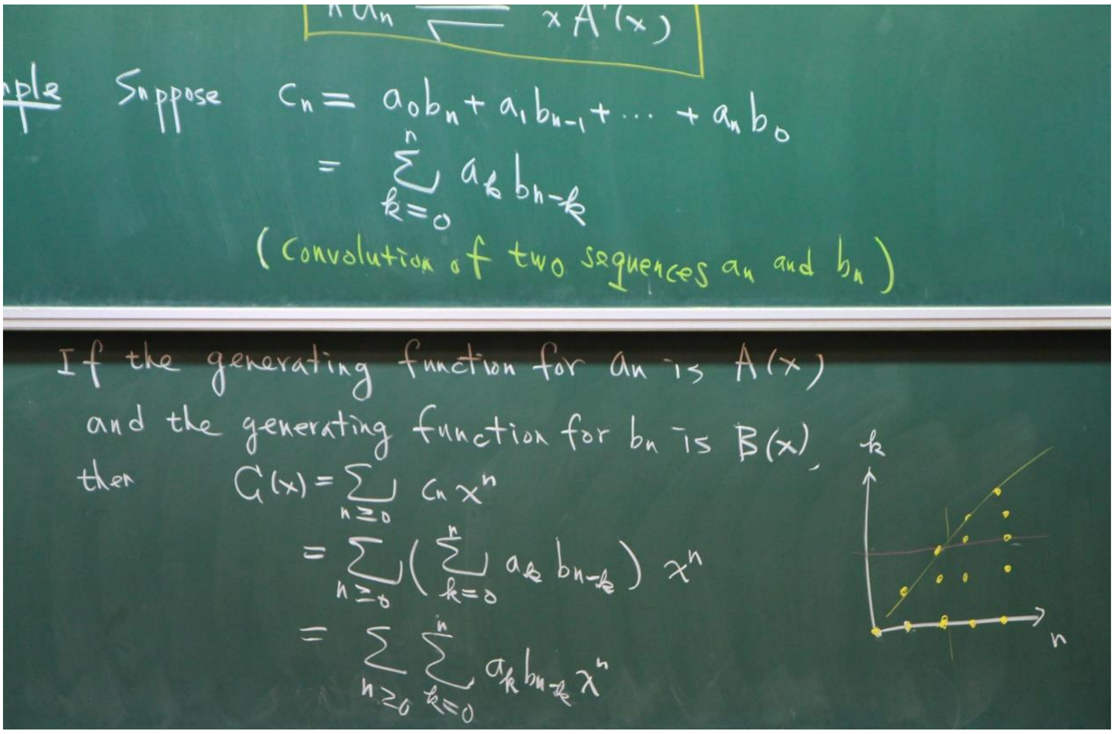


In general, $\boxed{\begin{array}{l} a_n \rightleftharpoons A(x) \\ n a_n \rightleftharpoons x A'(x) \end{array}}$

Example Suppose $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$

$$= \sum_{k=0}^{n} a_k b_{n-k}$$

(Convolution of two sequences a_n and b_n)



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If the generating function for a_n is $A(x)$

and the generating function for b_n is $B(x)$,

then
$$C(x) = \sum_{n\ge 0} c_n x^n$$
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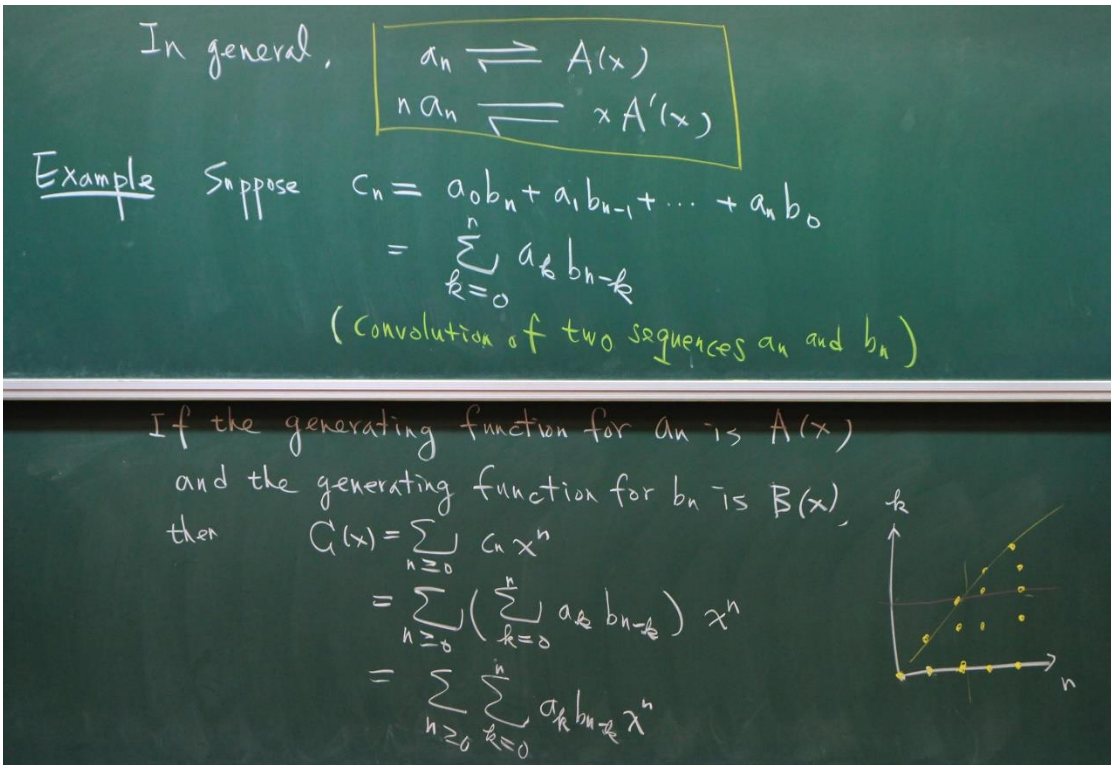

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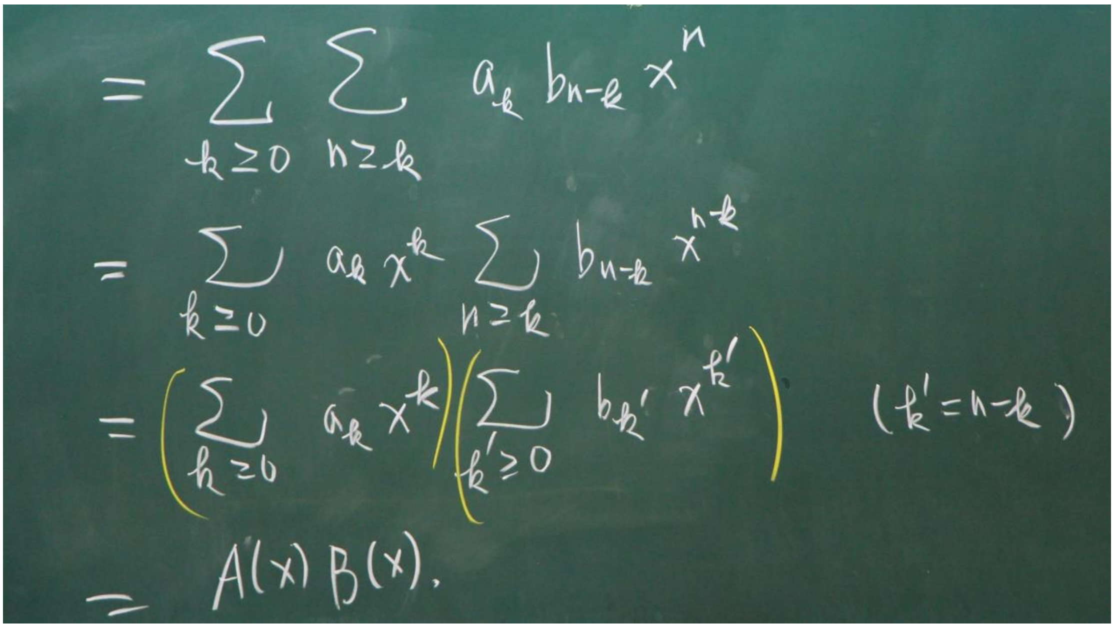

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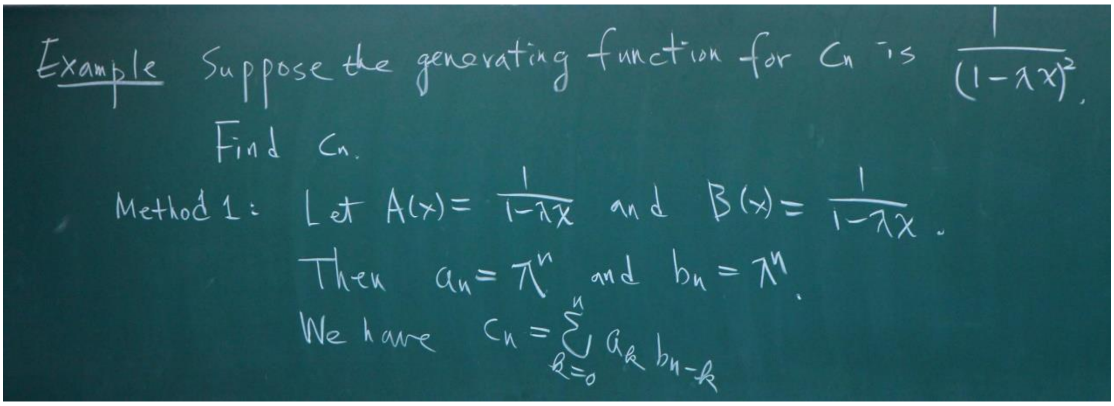
$$= \left(\sum_{k \ge 0} a_k x^k \right) \left(\sum_{k' \ge 0} b_{k'} x^{k'} \right) \qquad (k' = n-k)$$

$$= A(x) B(x).$$



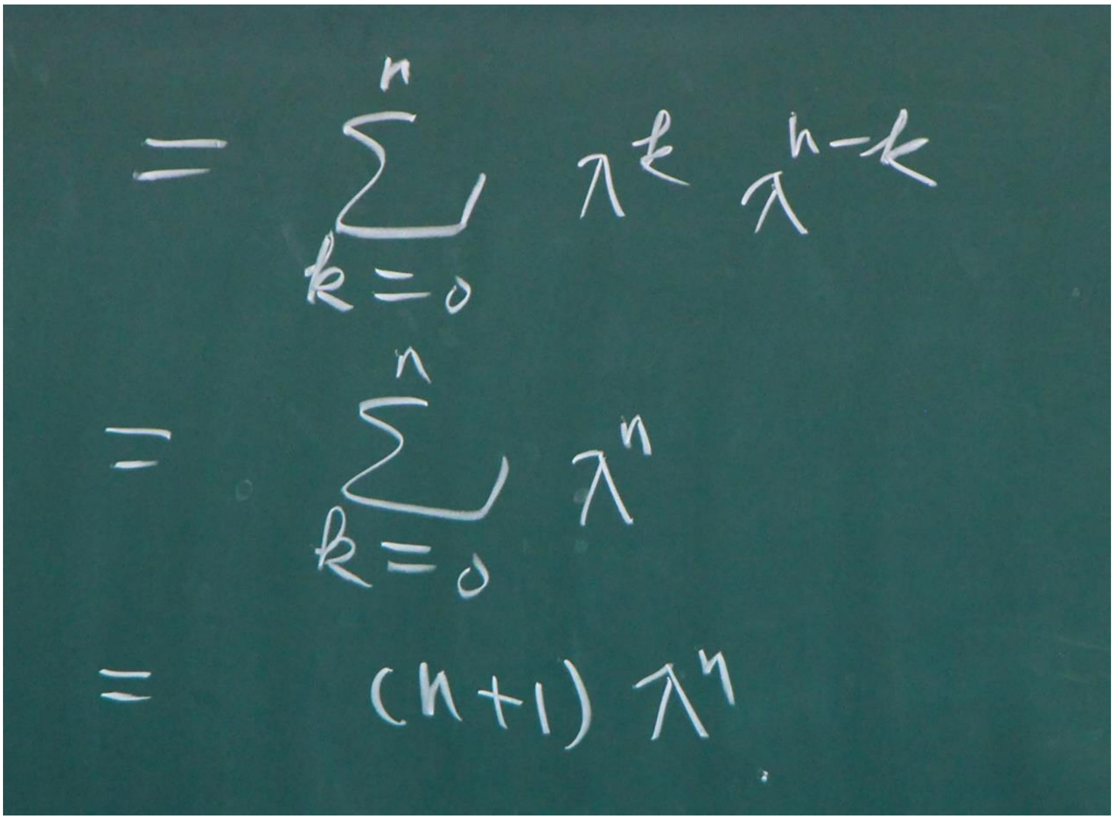
Example Suppose the generating function for c_n is $\frac{1}{(1-\lambda x)^2}$.

Find c_n.

Method 1: Let $A(x) = \frac{1}{1-\lambda x}$ and $B(x) = \frac{1}{1-\lambda x}$.

Then $a_n = \lambda^n$ and $b_n = \lambda^n$.

We have $c_n = \sum_{k=0}^{n} a_k b_{n-k}$



$$= \sum_{k=0}^{n} \lambda^k \lambda^{n-k}$$

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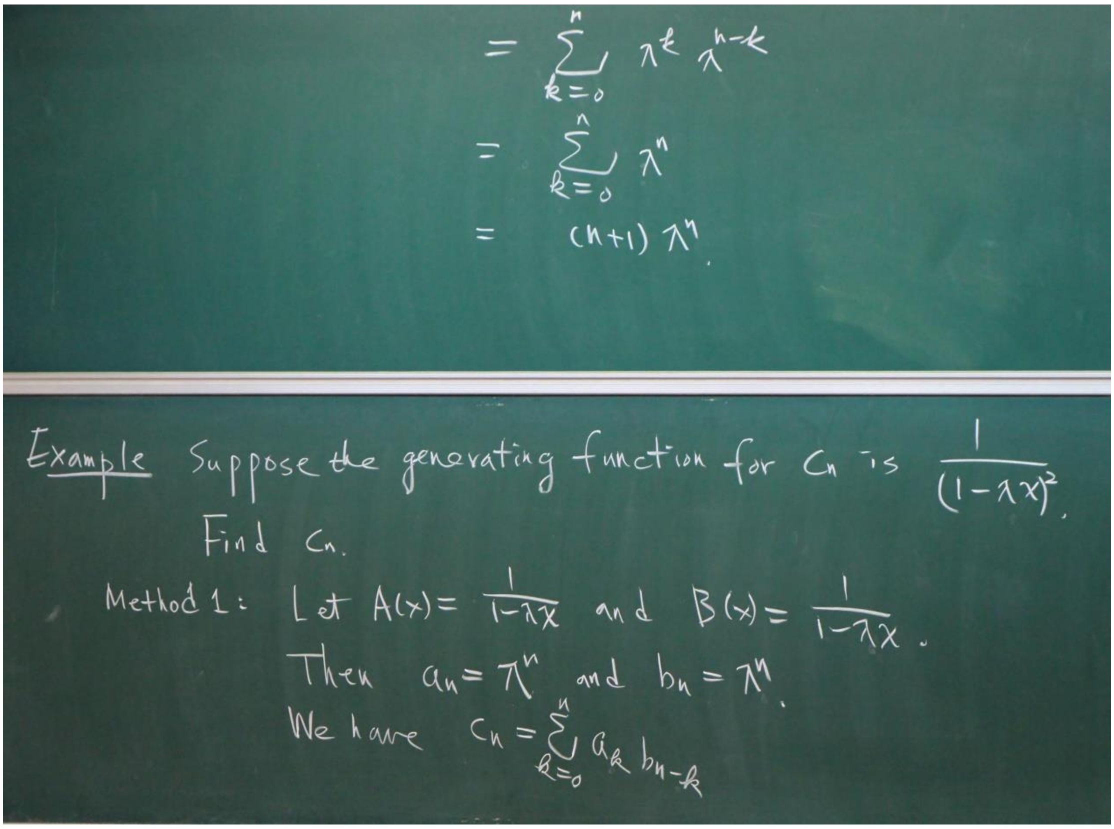
$$= (n+1)\lambda^n.$$



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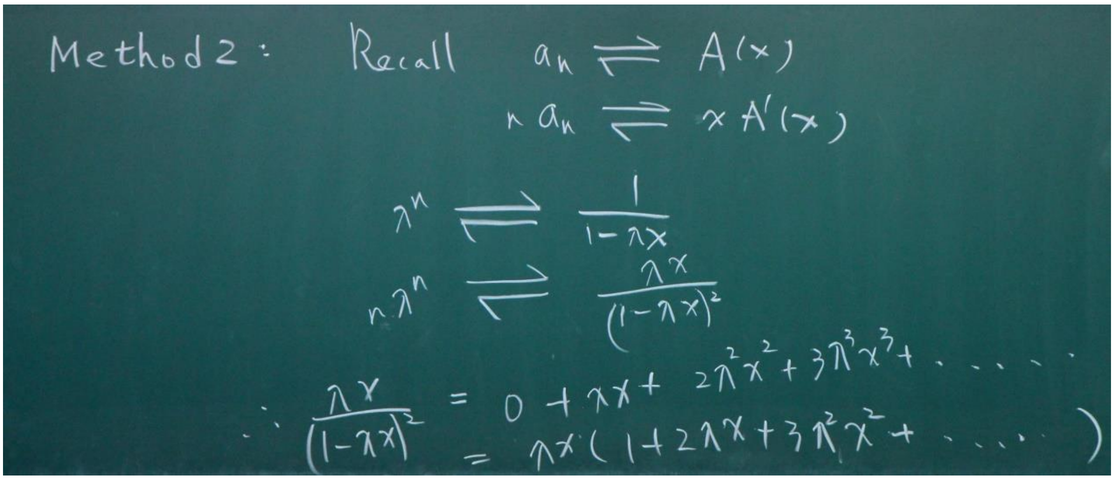


Method 2: Recall $a_n \rightleftharpoons A(x)$

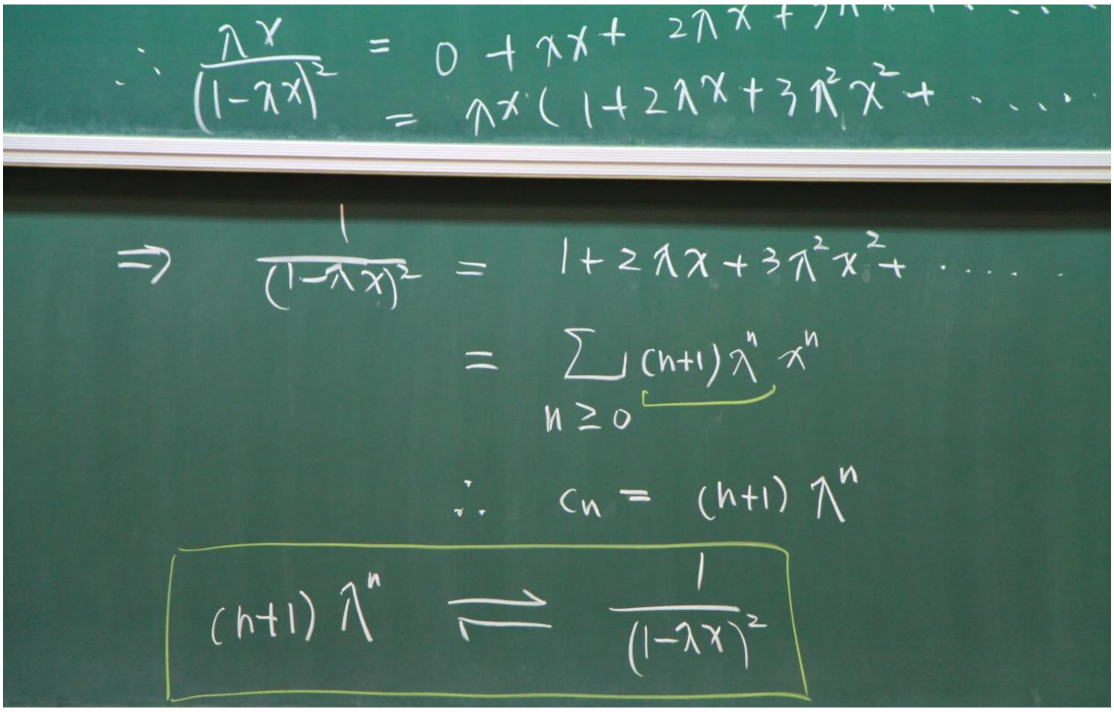
$$n a_n \rightleftharpoons x A'(x)$$

$$\lambda^n \rightleftharpoons \frac{1}{1-\lambda x}$$

$$n\lambda^n \rightleftharpoons \frac{\lambda x}{(1-\lambda x)^2}$$

$$\therefore \frac{\lambda x}{(1-\lambda x)^2} = 0 + \lambda x + 2\lambda^2 x^2 + 3\lambda^3 x^3 + \cdots\cdots$$

$$= \lambda x (1 + 2\lambda x + 3\lambda^2 x^2 + \cdots\cdots)$$



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$$\Rightarrow \frac{1}{(1-\lambda x)^2} = 1 + 2\lambda x + 3\lambda^2 x^2 + \cdots\cdots$$

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$$\therefore c_n = (n+1)\lambda^n$$

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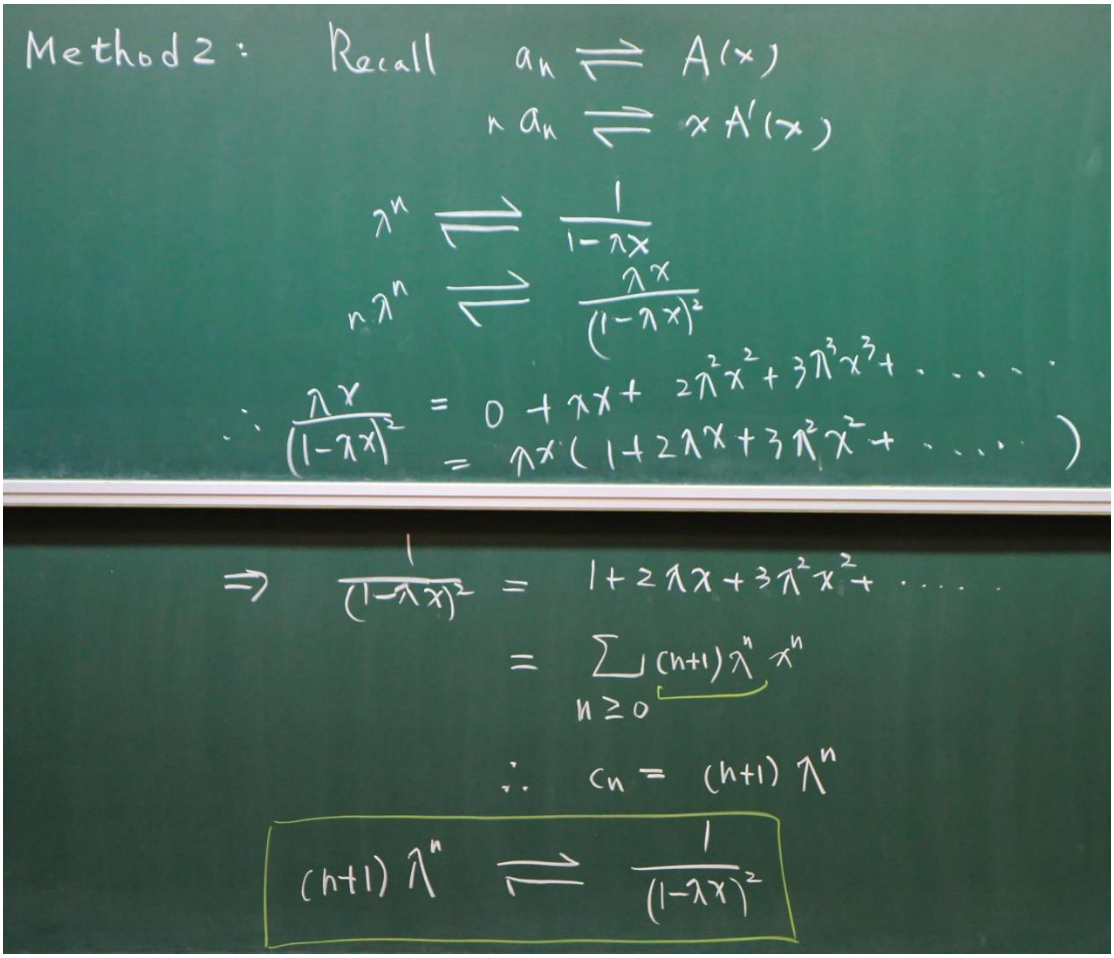

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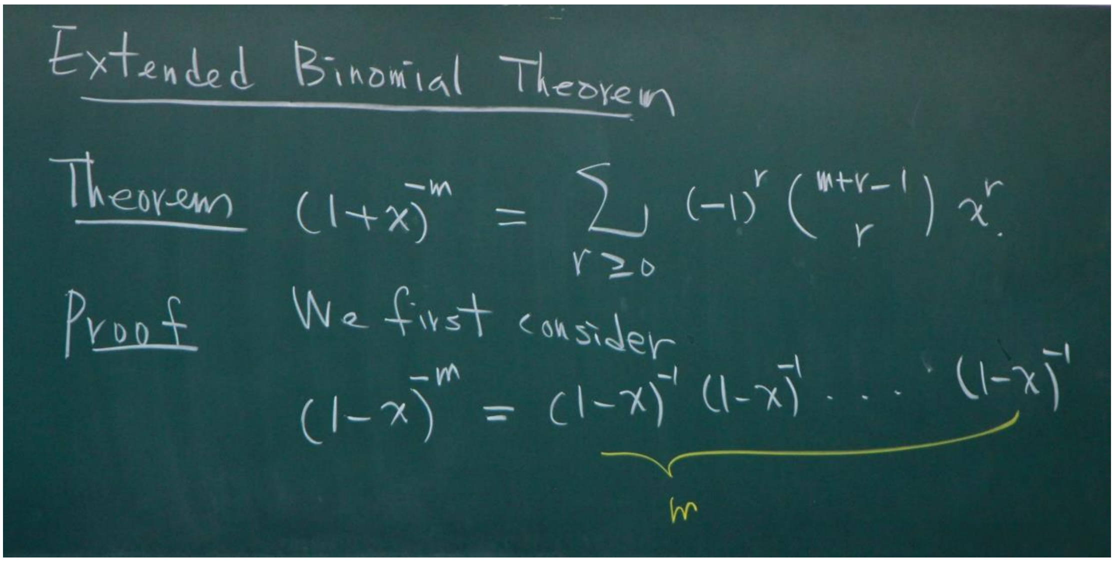
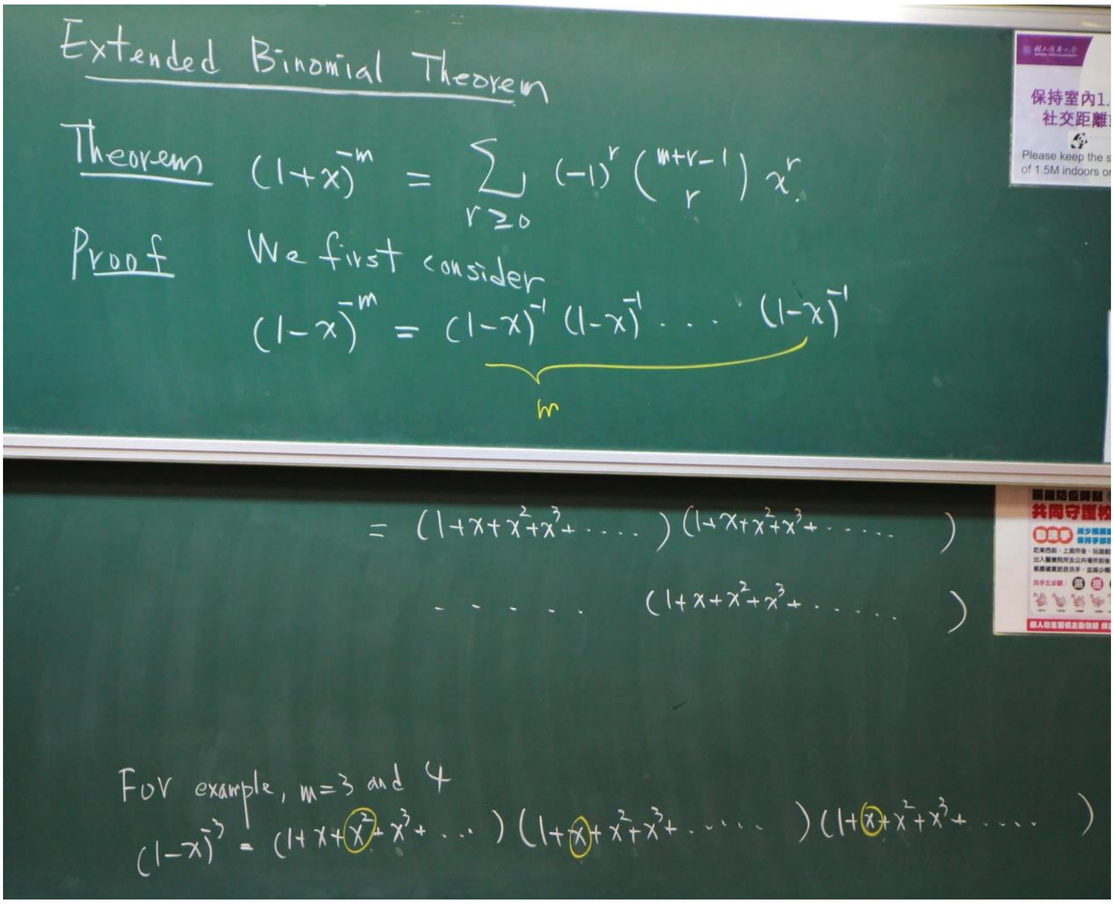


Extended Binomial Theorem

Theorem $(1+x)^{-m} = \sum_{r\ge 0} (-1)^r \binom{m+r-1}{r} x^r.$

Proof We first consider

$$(1-x)^{-m} = \underbrace{(1-x)^{-1}(1-x)^{-1}\cdots(1-x)^{-1}}_{m}$$



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$$= (1+x+x^2+x^3+\cdots\cdots)(1+x+x^2+x^3+\cdots\cdots)$$

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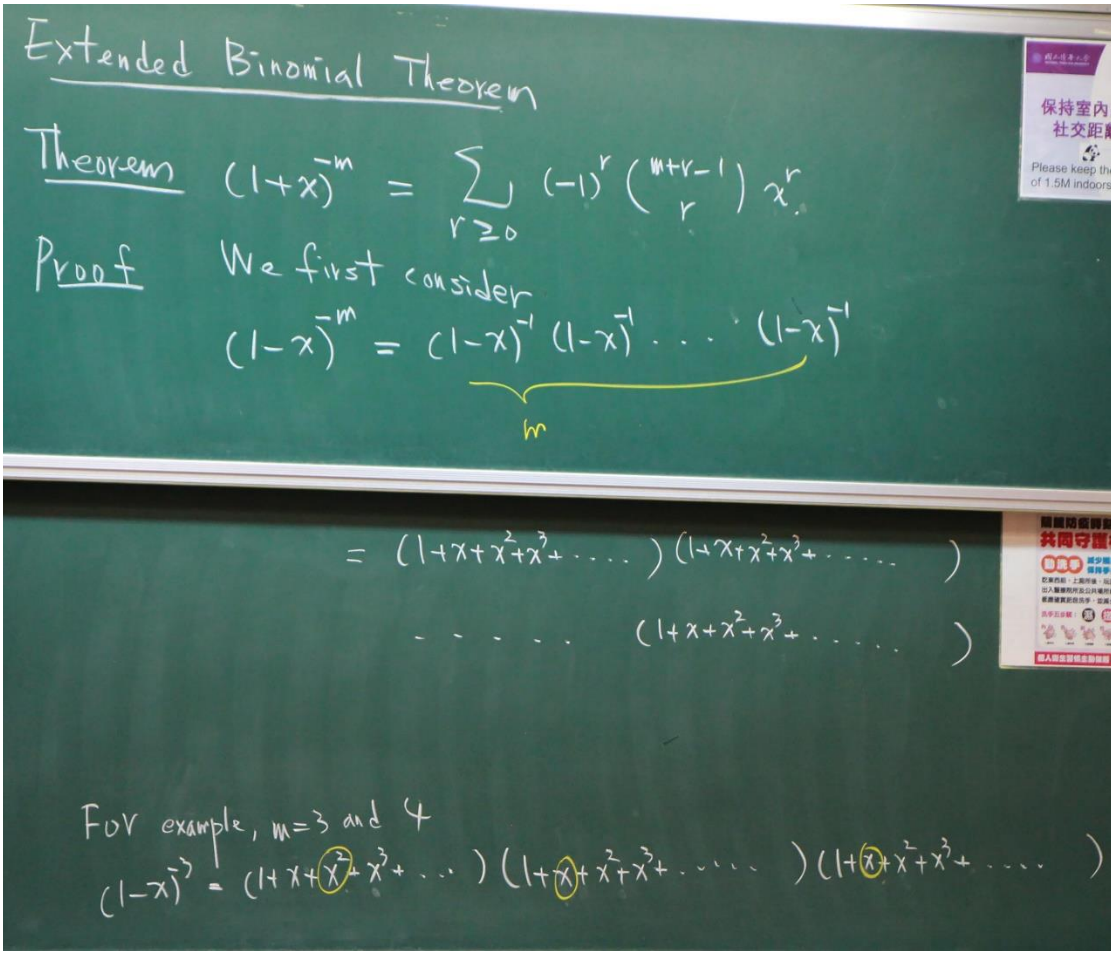
For example, m=3 and 4

$$(1-x)^{-3} = (1+x+\boxed{x^2}+x^3+\cdots)(1+\boxed{x}+x^2+x^3+\cdots\cdots)(1+\boxed{x}+x^2+x^3+\cdots\cdots)$$



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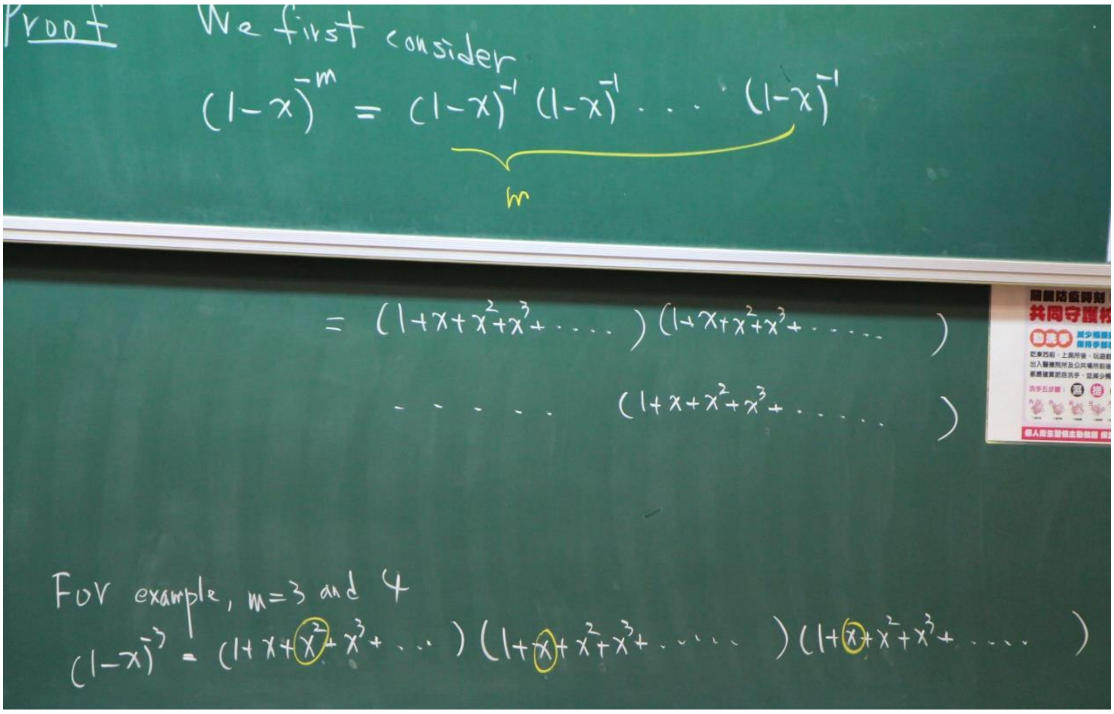

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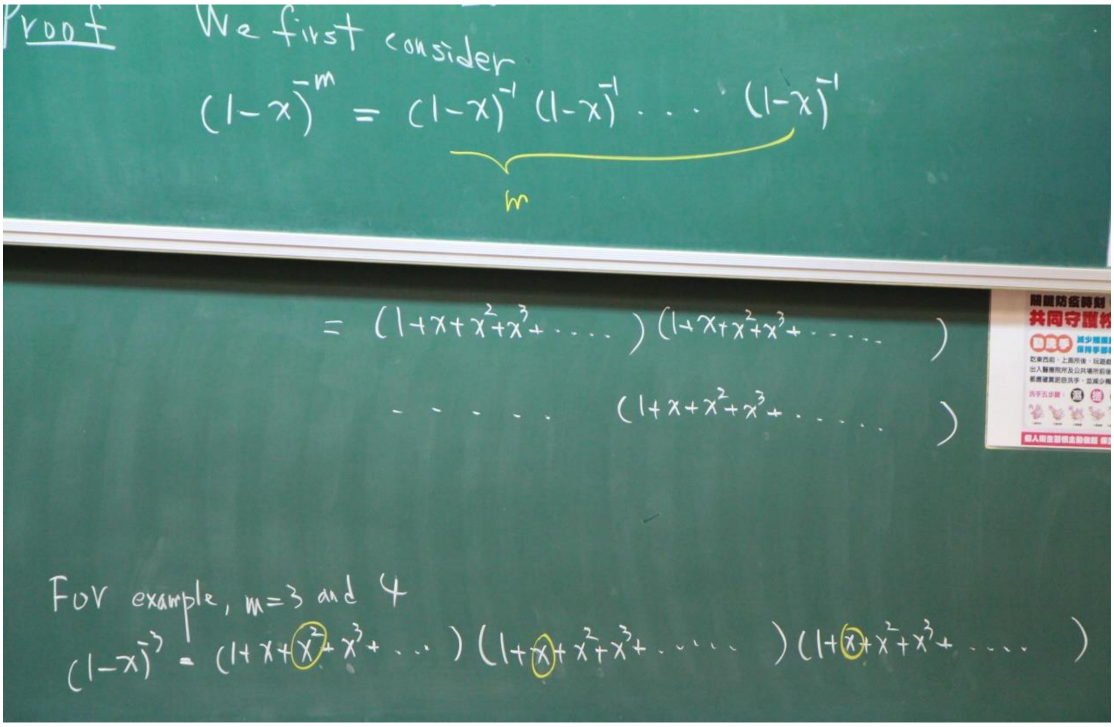

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$$= (1+x+x^2+x^3+\cdots\cdots)(1+x+x^2+x^3+\cdots\cdots)$$

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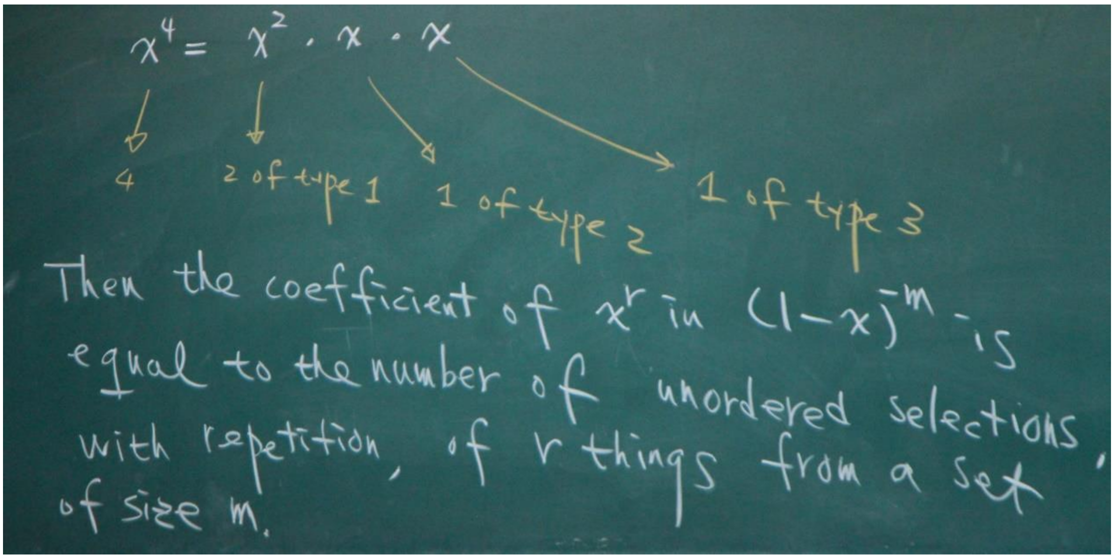
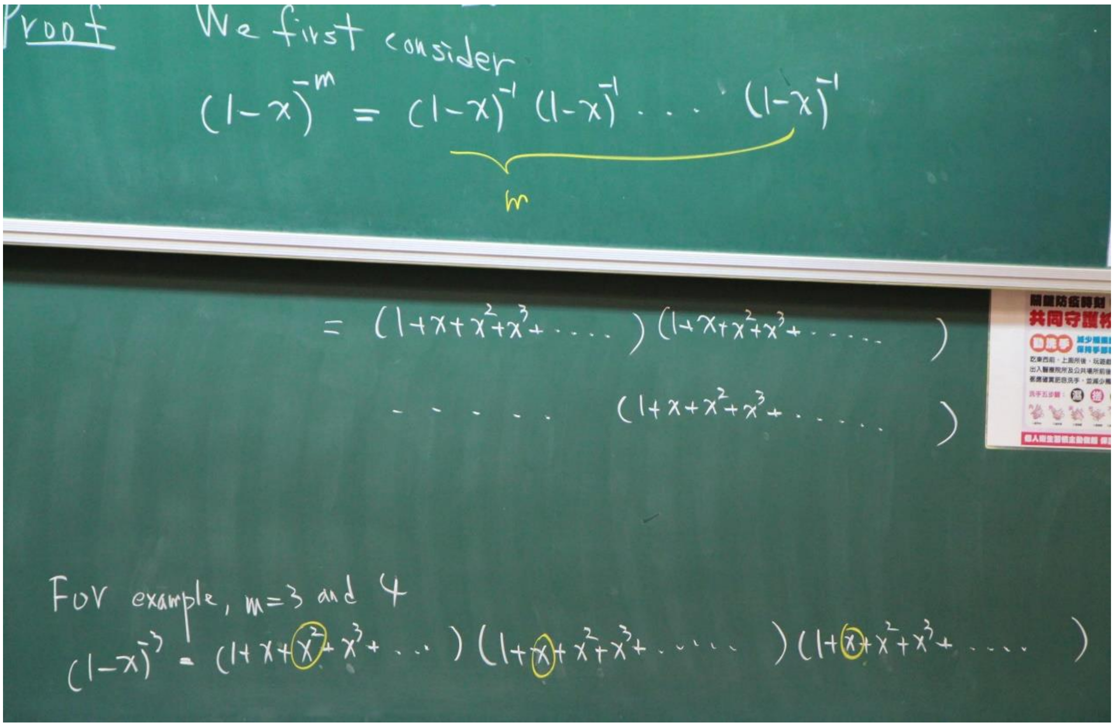
For example, m=3 and 4

$$(1-x)^{-3} = (1+x+\boxed{x^2}+x^3+\cdots)(1+\boxed{x}+x^2+x^3+\cdots\cdots)(1+\boxed{x}+x^2+x^3+\cdots\cdots)$$



$$\begin{tikzcd} x^4 \arrow[d] & = x^2 \arrow[d] & \cdot\; x \arrow[d] & \cdot\; x \arrow[d] \\ 4 & 2 \text{ of type } 1 & 1 \text{ of type } 2 & 1 \text{ of type } 3 \end{tikzcd}$$

Then the coefficient of x^r in $(1-x)^{-m}$ is equal to the number of unordered selections with repetition, of r things from a set, of size m.



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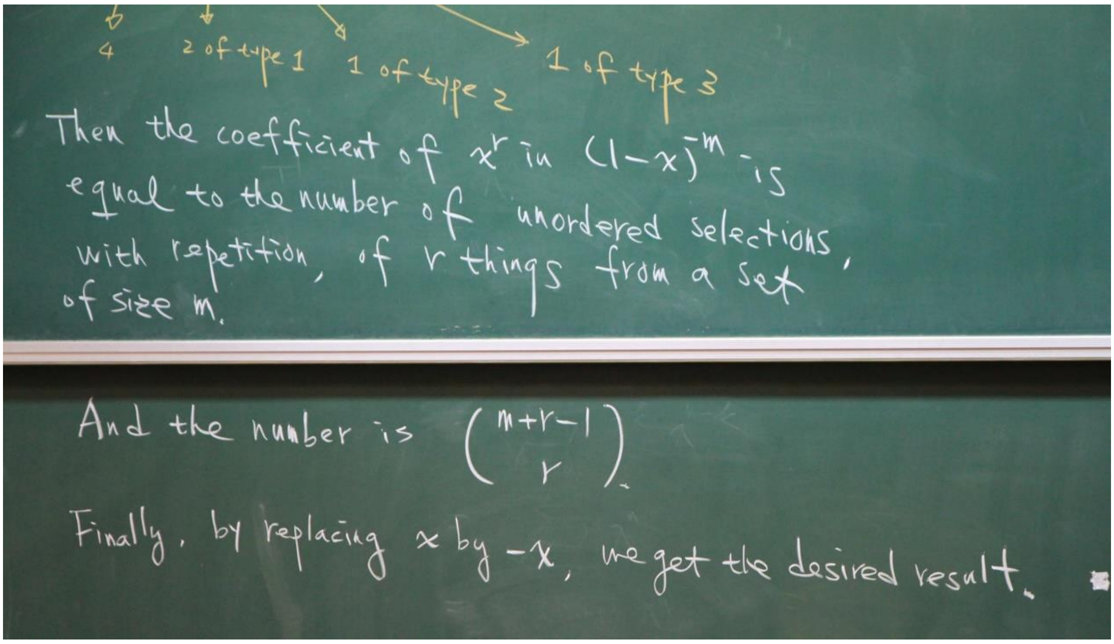


4

2 of type 1

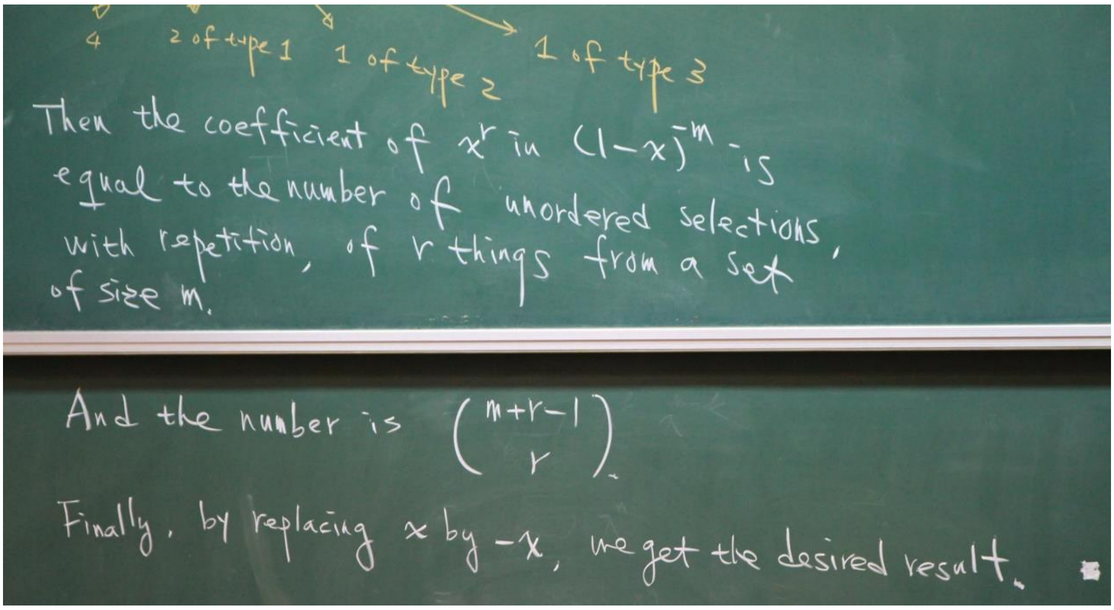
1 of type 2

1 of type 3

Then the coefficient of x^r in $(1-x)^{-m}$ is equal to the number of unordered selections, with repetition, of r things from a set of size m.

And the number is $\binom{m+r-1}{r}$.

Finally, by replacing x by $-x$, we get the desired result.



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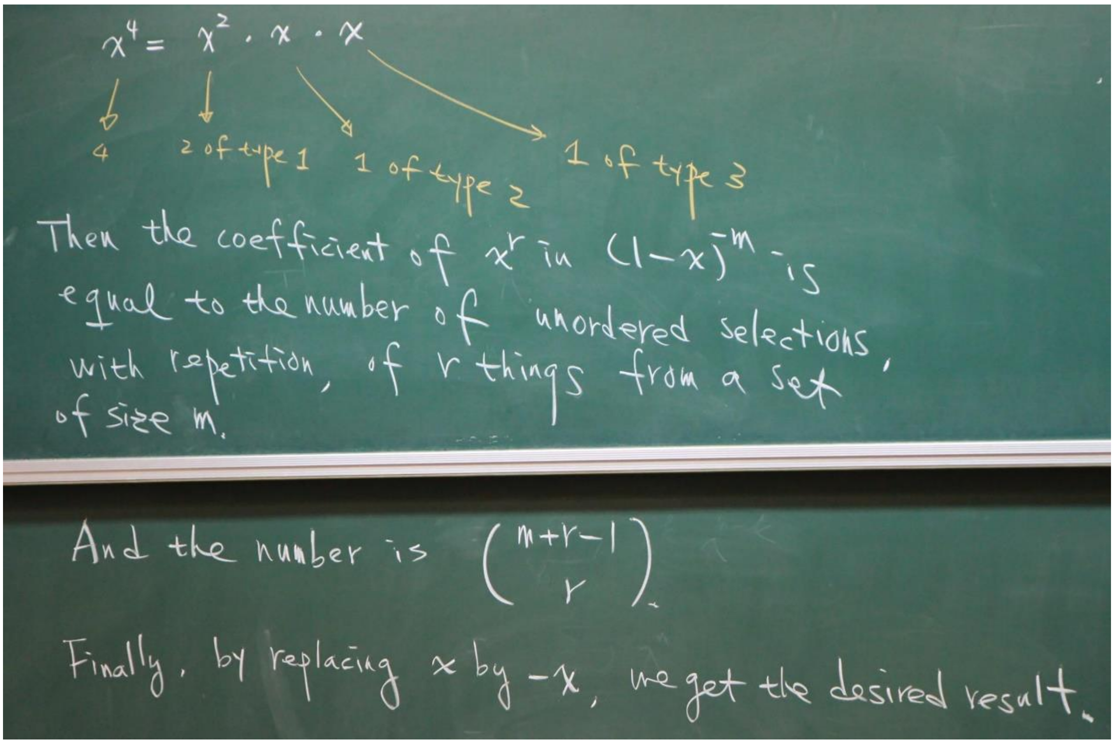

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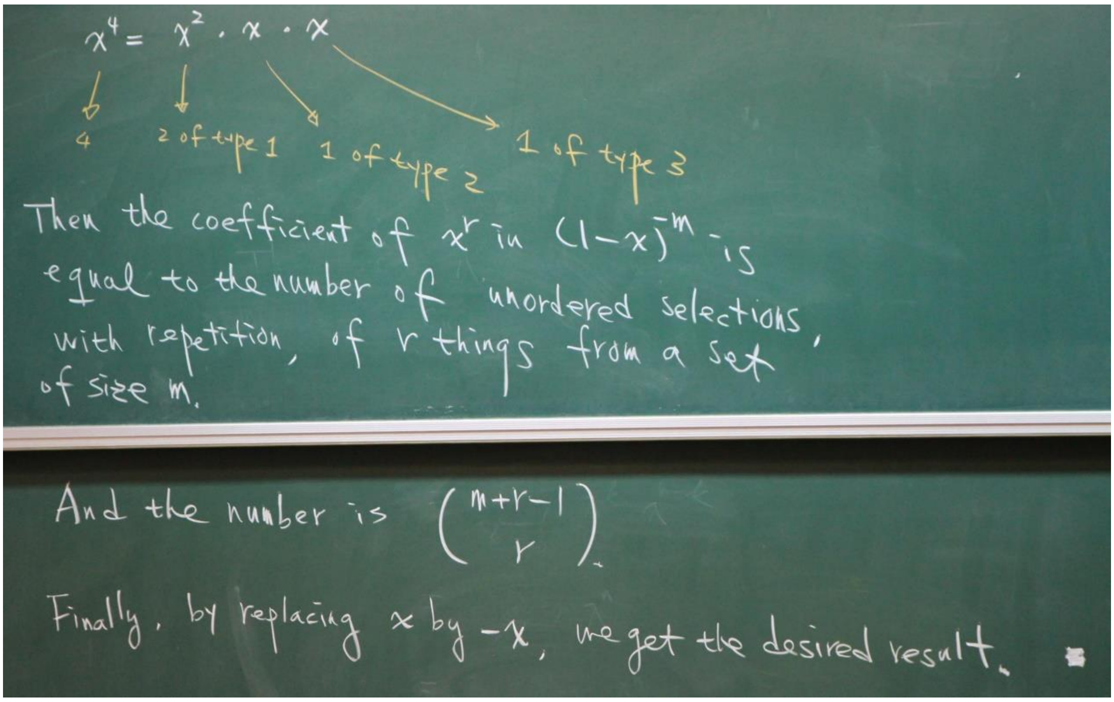

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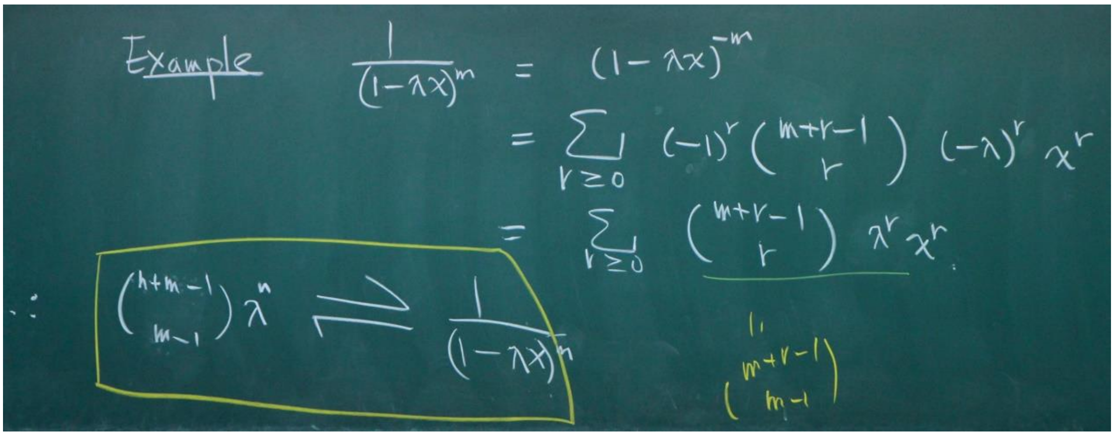


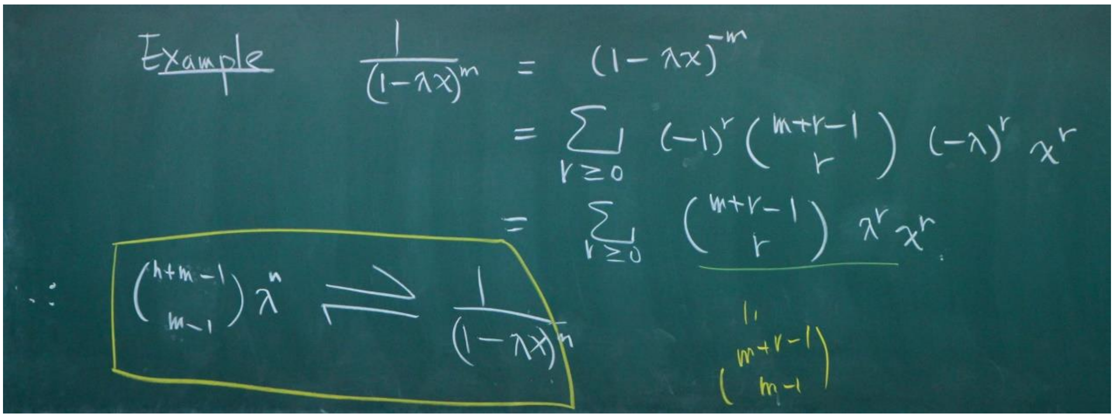
Example

$$\frac{1}{(1-\lambda x)^m} = (1-\lambda x)^{-m}$$

$$= \sum_{r\ge 0} (-1)^r \binom{m+r-1}{r} (-\lambda)^r x^r$$

$$= \sum_{r\ge 0} \underset{\binom{m+r-1}{m-1}}{\underset{\|}{\underline{\binom{m+r-1}{r} \lambda^r x^r}}}$$

$$\therefore \boxed{\binom{n+m-1}{m-1}\lambda^n \rightleftharpoons \frac{1}{(1-\lambda x)^m}}$$



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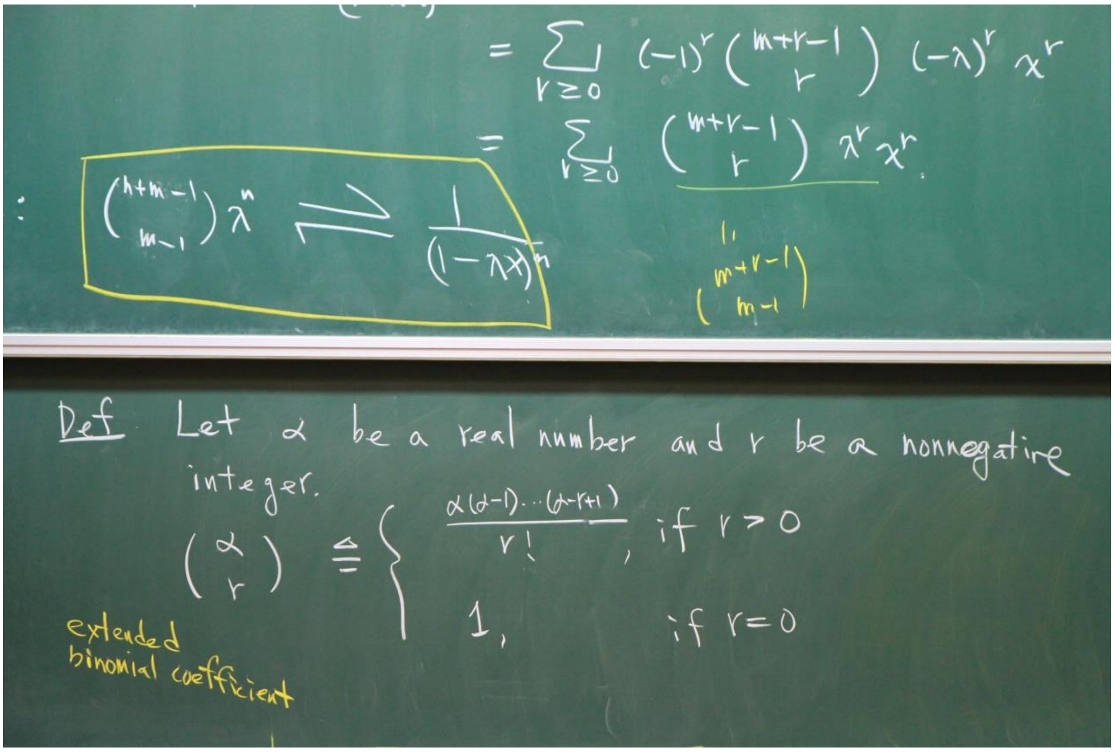

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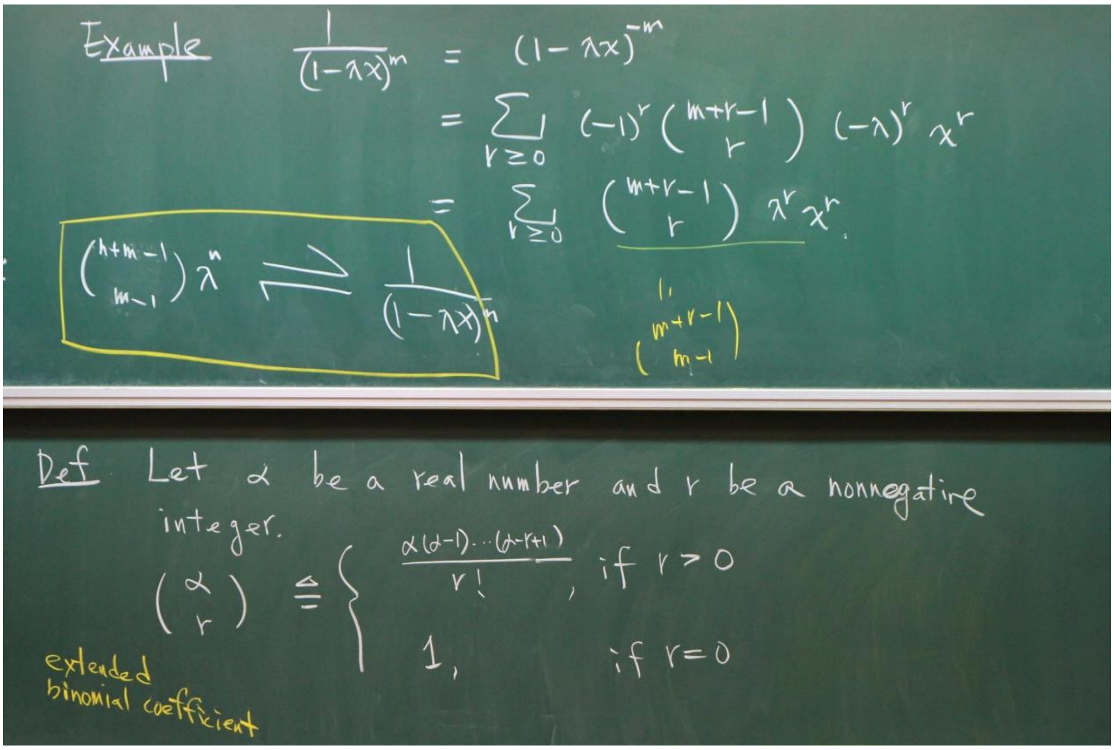
Def. Let α be a real number and r be a nonnegative integer.

$$\binom{\alpha}{r} \triangleq \begin{cases} \dfrac{\alpha(\alpha-1)\cdots(\alpha-r+1)}{r!}, & \text{if } r>0 \\ 1, & \text{if } r=0 \end{cases}$$

extended binomial coefficient



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$$(1-x)^{-m} = \underbrace{(1-x)^{-1}(1-x)^{-1}\cdots(1-x)^{-1}}_{m}$$

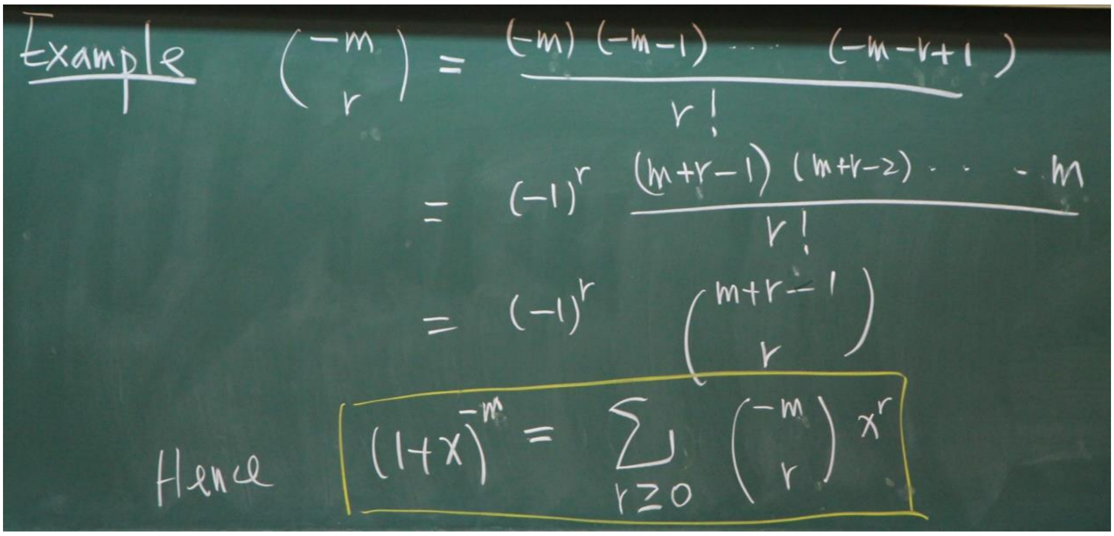
Example

$$\binom{-m}{r} = \frac{(-m)(-m-1)\cdots(-m-}{r!}$$

$$= (-1)^r \frac{(m+r-1)(m+r-}{r!}\ \ m$$

$$= (-1)^r \binom{m+r}{}$$

Hence $(1+x)^{-m}$ $\sum_{r\geq 0} \binom{-m}{r}$



Example

$$\binom{-m}{r} = \frac{(-m)(-m-1)\cdots(-m-r+1)}{r!}$$

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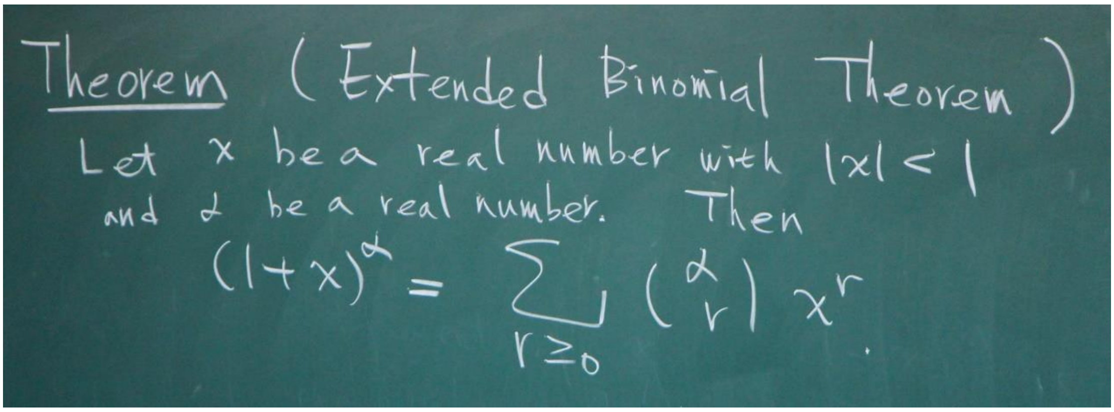
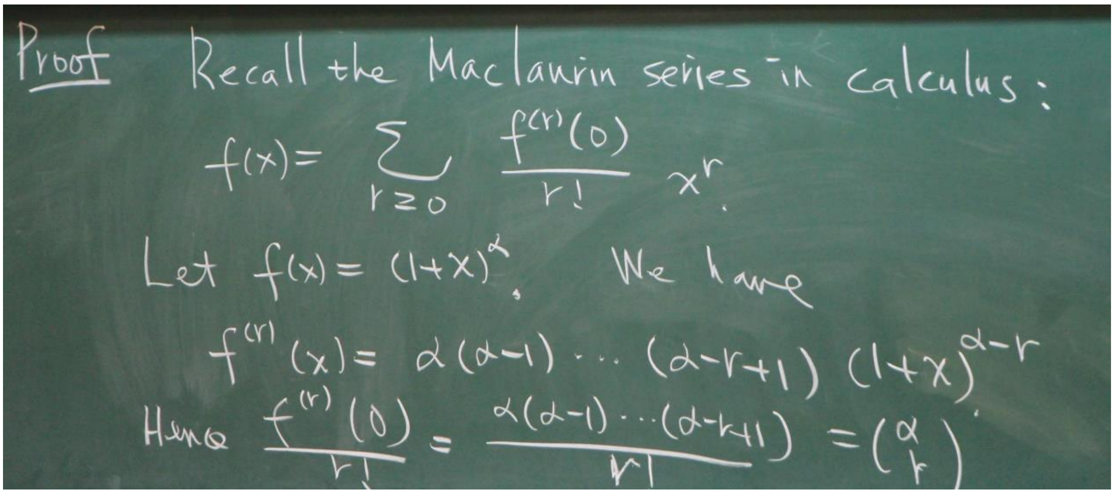


Theorem (Extended Binomial Theorem)

Let x be a real number with $|x|<1$ and α be a real number. Then

$$(1+x)^{\alpha} = \sum_{r\geq 0} \binom{\alpha}{r} x^r.$$



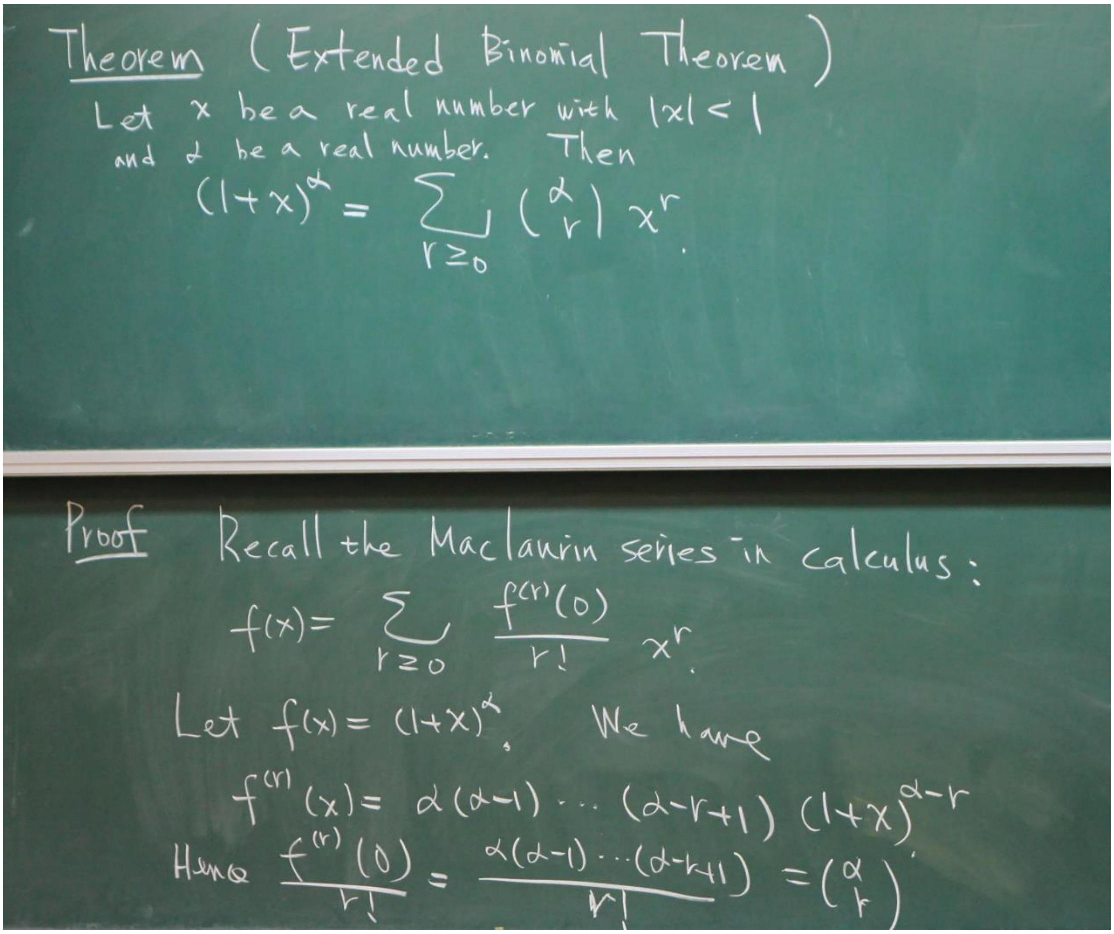
Proof Recall the Maclaurin series in calculus:

$$f(x) = \sum_{r \geq 0} \frac{f^{(r)}(0)}{r!} x^r.$$

Let $f(x) = (1+x)^\alpha$. We have

$$f^{(r)}(x) = \alpha(\alpha-1)\cdots(\alpha-r+1)(1+x)^{\alpha-r}$$

Hence $\frac{f^{(r)}(0)}{r!} = \frac{\alpha(\alpha-1)\cdots(\alpha-r+1)}{r!} = \binom{\alpha}{r}$.



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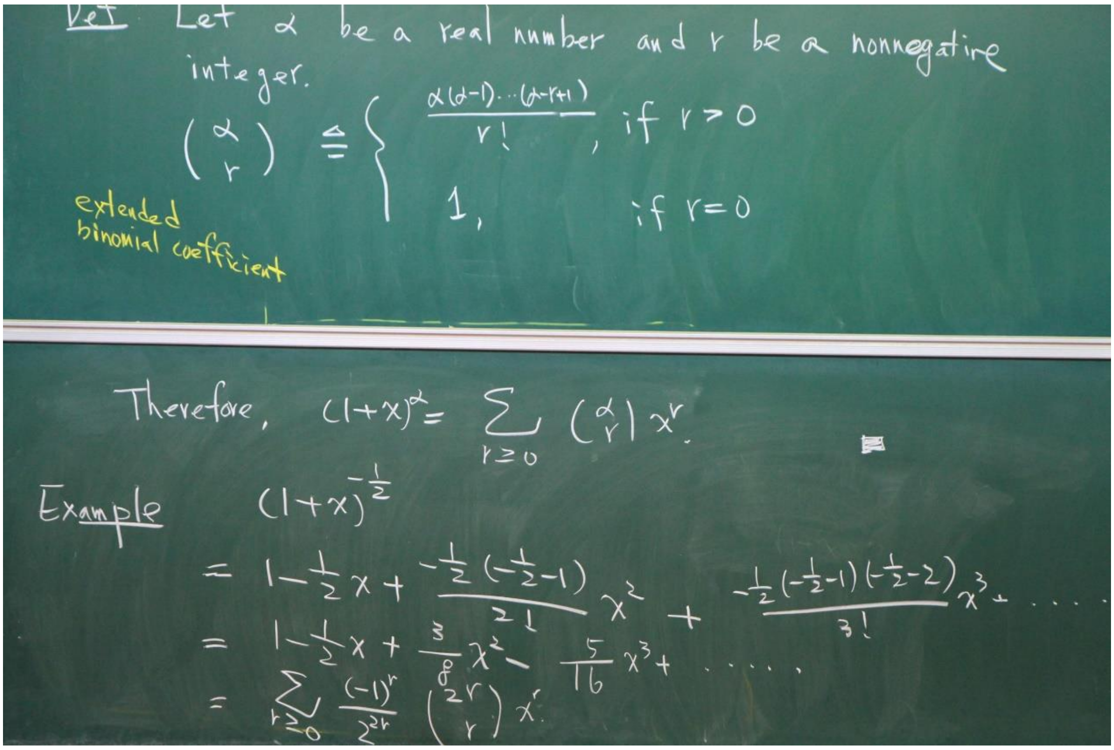

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extended binomial coefficient

Therefore, $(1+x)^{\alpha} = \sum_{r\geq 0} \binom{\alpha}{r} x^r$.

Example $(1+x)^{-\frac{1}{2}}$

$$= 1-\frac{1}{2}x + \frac{-\frac{1}{2}(-\frac{1}{2}-1)}{2!}x^2 + \frac{-\frac{1}{2}(-\frac{1}{2}-1)(-\frac{1}{2}-2)}{3!}x^3 + \cdots$$

$$= 1-\frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots$$

$$= \sum_{r\geq 0} \frac{(-1)^r}{2^{2r}} \binom{2r}{r} x^r$$